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Asymptotic Behavior of Densities for Two-Particle Annihilating Random Walks

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Consider the system of particles on $\mathbb{Z}^d$ where particles are of two types—A and B—and execute simple random walks in continuous time. Particles do not interact with their own type, but when an A-particle meets a B-particle, both disappear, i.e., are annihilated. This system serves as a model for the chemical reaction $A+B \to$ inert. We analyze the limiting behavior of the densities $\rho_A(t)$ and $\rho_B(t)$ when the initial state is given by homogeneous Poisson random fields. We prove that for equal initial densities $\rho_A(0)=\rho_B(0)$ there is a change in behavior from $d \leqslant 4$, where $\rho_A(t)=\rho_B(t) \sim C/t^{d/4}$, to $d \geqslant 4$, where $\rho_A(t)=\rho_B(t) \sim C/t$ as $t \to \infty$. For unequal initial densities $\rho_A(0)<\rho_B(0)$, $\rho_A(t) \sim e^{-C\sqrt{t}}$ in $d=1$, $\rho_A(t) \sim e^{-Ct/\log t}$ in $d=2$, and $\rho_A(t) \sim e^{-Ct}$ in $d \geqslant 3$. The term C depends on the initial densities and changes with d. Techniques are from interacting particle systems. The behavior for this two-particle annihilation process has similarities to those for coalescing random walks ($A+A \to A$) and annihilating random walks ($A+A \to$ inert). The analysis of the present process is made considerably more difficult by the lack of comparison with an attractive particle system.

KEY WORDS: Diffusion-dominated reaction; annihilating random walks; asymptotic densities; exact results.

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1. Introduction

Consider a system of particles of two types on $\mathbb{Z}^d$, A and B, which execute simple random walks in continuous time at rate 1. That is, the motion of different particles is independent and a particle at site x will jump to a given one of its $2d$ nearest neighbors at rate $1/2d$. Particles are assumed not to interact with their own type—multiple A particles or multiple B particles can occupy a given site. However, when a particle meets a particle of the opposite type, both disappear. (When a particle simultaneously meets more than one particle of the opposite type, it will only cause one of these particles to disappear.) We call this system a two-particle annihilating random walk.

One needs to specify an initial measure for the process. Two possibilities suggest themselves almost immediately. One can, on the one hand, independently throw down A and B particles according to the homogeneous Poisson random measures with probabilities

$$\begin{aligned} P[j \text{ type-}A \text{ particles at } x] &= e^{-r_A}(r_A)^j/j! \\ P[j \text{ type-}B \text{ particles at } x] &= e^{-r_B}(r_B)^j/j!; \end{aligned} \tag{1.1}$$

if there are initially both A and B particles at x, they immediately cancel each other out as much as possible. Another possibility is to assume that the initial state at each site is independent with a fixed probability of there being a single A particle, a single B particle, or no particle. Since our results (analyzing the system as $t \to \infty$) hold equally well for both initial measures (or for that matter, for anything "sufficiently ergodic"), we restrict ourselves for concreteness to the above Poisson random field construction. Associate with each A particle the value -1 and with each B particle the value 1. We denote by $\xi_t \in \mathbb{Z}^{\mathbb{Z}^d}$ the random state of the system at time t and by $\xi_t(x) \in \mathbb{Z}$ the signed number of particles at $x \in \mathbb{Z}^d$, i.e., $\xi_t(x) = (\#\, B \text{ particles at } x) - (\#\, A \text{ particles at } x)$. We can think of ξ_{0-} as the state before A-type and B-type particles initially at the same site have annihilated each other.

The two-particle annihilating random walk can serve as a model for the irreversible chemical reaction $A + B \to$ inert, where both particle types A and B are mobile. A and B can also represent matter and antimatter. There has been much interest in this model in the physics literature over the last several years following papers by Ovchinikov and Zeldovich[1] and Toussaint and Wilczek[2]; see Bramson and Lebowitz[3], where the results presented here were first announced, and refs. 4–8 for a more complete set of references. The main concern has been with the behavior of the densities in a spatially homogeneous system, i.e., with the expected number of A and B particles per site, say the origin,

$$\begin{aligned} \rho_A(t) &= E[\# A \text{ particles at } 0 \text{ at time } t] \\ \rho_B(t) &= E[\# B \text{ particles at } 0 \text{ at time } t], \end{aligned} \tag{1.2}$$

as $t \to \infty$. (The density of course does not depend on the site x.) The two basic cases are when (a) $0 < \rho_A(0) = \rho_B(0)$ (equal densities) and (b) $0 < \rho_A(0) < \rho_B(0)$ (unequal densities). Note that (a) corresponds to $0 < r_A = r_B$ and (b) to $0 < r_A < r_B$. Since $\rho_B(t) - \rho_A(t)$ must clearly remain constant for all t, one has $\rho_A(t) = \rho_B(t)$ in (a). Since

$$\lim_{t \to \infty} \rho_A(t) = 0 \tag{1.3}$$

will hold,

$$\lim_{t \to \infty} \rho_B(t) = \rho_B(0) - \rho_A(0) > 0 \tag{1.4}$$

in (b). The question then is at what rate the convergence in (1.3) holds. In case (a) there has been general agreement on the rate of convergence whereas in case (b) there have been many contradictory claims; refs. 7 contain the correct time asymptotics ($g_d(t)$ in (1.22)). Nowhere have we found the correct density dependence of the coefficients (ϕ_d in (1.23)). The results have at any rate not been rigorous from a mathematical point of view. It is the purpose of this article to provide such an approach. We start with some heuristics.

Equal Densities

For $\rho_A(0) = \rho_B(0)$, one can reason that $\rho_A(t)$ should decrease like $1/t^{d/4}$ for $d \leqslant 4$ and like $1/t$ for $d \geqslant 4$. The standard logic is that if one "neglects" the diffusive fluctuations in the number of the two types of particles present in a local region, as can be achieved physically by vigorous stirring, one can treat the positions of particles for the two types as being independent. The rate at which A particles meet B particles is then proportional to the density of each type present. This gives the "law of mass action" or mean field behavior,

$$\frac{d\rho_A(t)}{dt} = -k\rho_A(t)\,\rho_B(t) \tag{1.5}$$

for appropriate $k>0$. Since $\rho_A(t)=\rho_B(t)$, we have

$$\frac{d\rho_A(t)}{dt}=-k(\rho_A(t))^2, \tag{1.6}$$

and so

$$\rho_A(t)\approx 1/kt, \qquad \text{for large } t. \tag{1.7}$$

Throughout the article we shall use the following convention regarding "$\approx$" and "$\sim$": by $a(t)\approx b(t)$ we mean that $a(t)/b(t)\to 1$ as $t\to\infty$, whereas by $a(t)\sim b(t)$ we only mean that these functions are "close"—$a(t)/b(t)$ is of magnitude 1, or when appropriate, $\log a(t)/\log b(t)$ is of magnitude 1.

One can, on the other hand, also reason as follows. (This will be made precise in Section 2.) Let D_R denote the cube of side R which is centered at the origin. Also, let

$$\mathfrak{D}_R(t)=(\#B \text{ particles})-(\#A \text{ particles}) \text{ at time } t \text{ in } D_R. \tag{1.8}$$

We denote by η_t the stochastic process which behaves the same as ξ_t, except that particles merely execute random walks without interacting (annihilating) when meeting other particles. It seems reasonable to guess that

$$E[|\mathfrak{D}_R(t;\xi)-\mathfrak{D}_R(0;\xi)|]\sim E[|\mathfrak{D}_R(t;\eta)-\mathfrak{D}_R(0;\eta)|] \tag{1.9}$$

for large R. It is not difficult to show for $r_A=r_B$ that

$$E[|\mathfrak{D}_R(t;\eta)-\mathfrak{D}_R(0;\eta)|]\leqslant C_{1,d}\sqrt{r_A}\,R^{(d-1)/2}t^{1/4} \tag{1.10}$$

for appropriate $C_{1,d}$. If one believes (1.9) and (1.10), then

$$E[|\mathfrak{D}_R(t;\xi)-\mathfrak{D}_R(0;\xi)|]\leqslant C_{1,d}\sqrt{r_A}\,R^{(d-1)/2}t^{1/4}. \tag{1.11}$$

But for $r_A=r_B$,

$$E[|\mathfrak{D}_R(0;\xi)|]\geqslant C_{2,d}\sqrt{r_A}\,R^{d/2} \tag{1.12}$$

for appropriate constants $C_{2,d}$. That is, there is a local fluctuation in the numbers of the A and B particles. By (1.11) and (1.12),

$$E[|\mathfrak{D}_R(t;\xi)|]\geqslant C_{2,d}\sqrt{r_A}\,R^{d/2}-C_{1,d}\sqrt{r_A}\,R^{(d-1)}t^{1/4}. \tag{1.13}$$

Now choose R at time t to be $R_t=a\sqrt{t}$. For a large enough, (1.13) is at least $b\sqrt{r_A}\,R_t^{d/2}$ for some $b>0$. By symmetry,

$$\rho_A(t)\geqslant \tfrac{1}{2}R_t^{-d}E[|\mathfrak{D}_{R_t}(t;\xi)|]. \tag{1.14}$$

Plugging in the bound for $\mathfrak{D}_{R_t}(t;\xi)$ and substituting for R_t, we obtain

$$\rho_A(t) \geqslant c\sqrt{r_A}/t^{d/4}, \tag{1.15}$$

with $c = b/2a^{d/2}$.

One needs to reconcile (1.15) with (1.7). The standard heuristics are that the term (1.15) measuring local fluctuations dominates in $d<4$, whereas the mean field limit in (1.7) is accurate for $d \geqslant 4$. The densities $\rho_A(t)$ and $\rho_B(t)$ should therefore decay asymptotically like $t^{-d/4}$ for $d \leqslant 4$ and t^{-1} for $d \geqslant 4$. Our first result verifies this behavior.

Theorem 1. Assume that $r_A = r_B > 0$ with the initial measures given as in (1.1). There exist positive constants c_d and C_d such that

$$\begin{aligned} c_d\sqrt{r_A}/t^{d/4} &\leqslant \rho_A(t) = \rho_B(t) \leqslant C_d\sqrt{r_A}/t^{d/4}, && d<4, \\ c_4(\sqrt{r_A} \vee 1)/t &\leqslant \rho_A(t) = \rho_B(t) \leqslant C_4(\sqrt{r_A} \vee 1)/t, && d=4, \\ c_d/t &\leqslant \rho_A(t) = \rho_B(t) \leqslant C_d/t, && d>4, \end{aligned} \tag{1.16}$$

for large enough t.

Presumably, $t^{d/4}\rho_A(t)$ in $d \leqslant 4$ and $t\rho_A(t)$ in $d \geqslant 4$ have limits as $t \to \infty$, although our techniques do not show this.

The asymptotic densities given here share certain similarities in common with those for two related simpler models. As done here, one can define a process consisting of particles on $\mathbb{Z}^d$ which execute independent simple random walks except when two particles attempt to occupy the same site. We now assume, however, that there is only one type of particle (say A), and that when two particles attempt to occupy the same site either (a) they coalesce into one particle and afterward behave as just one particle, or (b) they annihilate one another. The first model can be interpreted as the chemical reaction $A + A \to A$, and is called coalescing random walk, while the second model corresponds to $A + A \to$ inert, and is called annihilating random walk. For each of these models at most one particle is permitted per site. It is most natural to consider the state where all sites are occupied as the initial state although the same limiting behavior holds for a much larger class of states.

The coalescing random walk is attractive. This says, basically, that adding more particles to the system initially will not diminish the number of particles later on. It is also the dual of the voter model. (Liggett[9] is the most complete general reference on interacting particle systems. Griffeath[10] and Durrett[11] are also useful references and emphasize the role of duality.) For these reasons, it is possible to analyze the density $\rho(t)$ and show that:

$$\rho(t) \approx \frac{1}{\sqrt{\pi t}}, \qquad d=1,$$
$$\approx \frac{1}{\pi}\frac{\log t}{t}, \qquad d=2, \tag{1.17}$$
$$\approx \frac{1}{\gamma_d t}, \qquad d \geqslant 3,$$

for appropriate γ_d. The case $d=1$ is easy and is an application of the above duality and the local central limit theorem. For $d \geqslant 2$, see Bramson and Griffeath[12]. The annihilating random walk can, it turns out (Arratia[13]), be compared directly to the coalescing random walk. Let $\tilde{\rho}(t)$ denote its density. Since $\tilde{\rho}(t)/\rho(t) \to 1/2$ as $t \to \infty$, one has

$$\tilde{\rho}(t) \approx \tfrac{1}{2}\rho(t). \tag{1.18}$$

Note that for coalescing and annihilating random walk, $d=2$ is where the crossover in asymptotic behavior $\rho(t)$, $\tilde{\rho}(t)$ occurs, rather than at $d=4$ as found here for $A+B \to$ inert. This is connected in the first case with the recurrence of random walk in $d \leqslant 2$ and its transience in $d>2$.

Unequal Densities

For $\rho_A(0) < \rho_B(0)$, the asymptotic behavior of $\rho_A(t)$ should be quite different. Since $\lim_{t\to\infty} \rho_B(t) = \rho_B(0) - \rho_A(0) > 0$, there is always at least density $b = \rho_B(0) - \rho_A(0) > 0$ of type B particles in the population. The density $\rho_A(t)$ must therefore decrease much more rapidly than if $\rho_A(0) = \rho_B(0)$. From (1.5), one would obtain

$$\frac{d\rho_A(t)}{dt} = -k(b + o(1))\,\rho_A(t). \tag{1.19}$$

Consequently, one might expect that

$$\rho_A(t) = \rho_A(0)e^{-k(b+o(1))t}. \tag{1.20}$$

On the other hand, as in the case $\rho_A(0) = \rho_B(0)$, local fluctuations could conceivably alter the relative proportions of type A and type B particles locally, and cause a different rate of decay. Presumably, as before, this change would be associated with lower dimensions. There are various different conclusions in the physics literature. Here, we show the following:

Theorem 2. Assume that $0 < r_A < r_B$ with the initial measures given as in (1.1). There exist positive constants A_d and λ_d such that

$$\exp[-A_d \phi_d g_d(t)] \leqslant \rho_A(t) \leqslant \exp[-\lambda_d \phi_d g_d(t)] \tag{1.21}$$

for large enough t, where

$$\begin{aligned} g_d(t) &= \sqrt{t}, && d=1, \\ &= t/\log t, && d=2, \\ &= t, && d \geqslant 3, \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} \phi_d &= (r_B - r_A)^2/r_B, && d=1, \\ &= r_B - r_A, && d \geqslant 2. \end{aligned} \tag{1.23}$$

The mean field limit is thus valid for $d \geqslant 3$, but not in $d = 1, 2$. The dependence on initial densities is different in $d = 1$ than that in $d > 1$, which corresponds to (1.20). The reason is the presence of greater fluctuations in $d = 1$.

The methodology employed for Theorems 1 and 2 involves in both cases different estimates for upper and lower bounds. Lower bounds for $r_A = r_B$ in $d \leqslant 4$ are obtained in Section 2 (Proposition 1). The reasoning follows the outline given in (1.9)–(1.15) and is quite simple. The mean field lower bounds in (1.7) turn out (unexpectedly) to be much trickier; these are given in Section 3 (Proposition 2). The upper bounds for $r_A = r_B$ are derived in Section 4 (Proposition 3). Together with certain estimates from Section 2, they involve amplification of a technique introduced in ref. 12. The behavior for $r_A < r_B$ is given in Sections 5, 6, and 7. The derivation of the lower bound in Section 5 is relatively easy if one neglects the dependence on r_A, r_B, but requires more preparation as formulated in (1.21)–(1.23) (Proposition 4). The derivation of the upper bound takes some time and is given in Section 6 for $d > 1$ (Proposition 5) and in Section 7 for $d = 1$ (Proposition 6).

Some Formalism

The type of stochastic process considered here can be rigorously constructed by a slight modification of the standard framework of interacting particle systems[10]. It is a continuous time Markov process whose set of states is $\mathbb{Z}^{\mathbb{Z}^d}$—a positive value at a site $x \in \mathbb{Z}^d$ denotes the presence of B particles, whereas a negative value the presence of A particles. If there are

j type-B particles at x, we can think of them as inhabiting "levels" $l=1, 2,..., j$, whereas if there are j type-A particles at x, we can think of them as inhabiting levels $l=-1, -2,..., -j$. The evolution of the system of particles can be completely specified by a *percolation substructure* $\mathscr{P}$. At each site $x \in \mathbb{Z}^d$ one can construct a family of independent exponential mean-1 random variables $W_{x,l}(k)$, $l \in \mathbb{Z}$, $k \in \mathbb{Z}^+$. The random variables $T_{x,l}(k) = \sum_{i=1}^k W_{x,l}(i)$, $k \in \mathbb{Z}^+$, are to be the times at which "alarm clocks" go off. At each such time $T_{x,l}(k)$, an arrow is laid down which points from x to one of its $2d$ immediate neighbors, each being chosen with probability $1/2d$; the particle at level l at position x, if there is one, jumps in the direction specified by the accompanying arrow. There may be, after a jump, more than one particle at a given level at a site; there may also be "holes," with lower levels not being occupied (either due to annihilation or a vacancy). After each particle moves, we therefore reorder the particles at both the former and target sites to fill up levels in the order specified above. For example, suppose that at site x the levels 1–4 are occupied when the alarm clock at level 2 goes off. Since the departure of the B particle at level 2 creates a hole at x, the B particles at levels 3 and 4 are immediately reassigned to levels 2 and 3. If the target site had particles at levels 1 and 2, it will afterwards have particles at levels 1–3. If it had particles at levels -1 and -2, then one of these A particles will be annihilated by the arriving B particle, and the site will afterwards have a particle at -1. This procedure defines the evolution of the system for all time. It will be utilized at various points in the article. More detail on the corresponding percolation substructure with only one level is given in refs. 10 and 11.

The probability space Ω can be given concretely by $\mathscr{P}$ and the initial configuration on $\mathbb{Z}^d$. (Occasionally, such as in Section 3, we will enlarge the space slightly by including additional information at $t=0$.) $\mathscr{F}_t$ will stand for the σ-algebra generated by ξ_{0-} and by the percolation substructure up to time t; $\mathscr{F}_t^\xi$ for the σ-algebra generated by ξ_s, $0 \leqslant s \leqslant t$; and $\mathscr{F}_\infty$, $\mathscr{F}_\infty^\xi$ for the σ-algebras at $t=\infty$. Clearly, $\xi_t \in \mathscr{F}_t^\xi \subset \mathscr{F}_t$.

2. Lower Bounds for Equal Densities, $d \leqslant 4$

In this section we will show that $\rho_A(t)$ decays at most like $c_d \sqrt{r_A}/t^{d/4}$ for all dimensions d if $r_A = r_B$. For $d \leqslant 4$, this will provide us with the correct lower bound. The proof is not difficult and follows from three elementary lemmas which will also be useful in the proof of the upper bound. As sketched in (1.9)–(1.15), the basic point is that one can compare our process ξ with the process η having the same percolation substructure as ξ, but where particles execute random walks which do not interact. This is done in Lemma 2.1. In Lemma 2.2 we measure how much η can change up to time t. Lemma 2.3 gives a simple estimate on ξ_0. The proof of Proposition 1 puts these results together.

Here and in later sections, we will need to introduce various constants for our calculations. When these values are not important, we shall label them as, for example, C_1, C_2, C_3,... or c_1, c_2,.... When there is no chance of confusion, these constants will be "recycled" in different sections. Also, although not stated explicitly, these constants will be allowed to depend on d. To explicitly exhibit dependence on d we shall use the notation $C_{k,d}$ and $c_{k,d}$. We also mention here that we will generally ignore the fact that the number of lattice sites in the various cubes D_R of side R we shall use depends discontinuously on R; the number of lattice sites also varies somewhat for translates of D_R. All we need is that for R not too small, this number differs from R^d by at most a constant factor, which we will suppress.

Lemma 2.1 states that the average value of any convex function φ of the difference of the numbers of A and B particles in D_R at time t will be at least as large for η as for ξ. The basic reason is that pairs of A and B particles in ξ which annihilate can be thought of as being forced to remain together forever; this is consistent with annihilation. Consequently, after their collision both or neither is in D_R. For η, one more or one less net particle may be in D_R (A-type $= -1$ or B-type $= 1$) since these particles do not remain together. By symmetry, either outcome is equally likely. The lemma will follow from this and the convexity of φ.

Lemma 2.1. Set

$$\hat{\mathfrak{D}}_R(t;\cdot) = \mathfrak{D}_R(t;\cdot) - X,$$

where the random variable X is independent of the percolation substructure $\mathscr{P}$ of ξ (and hence also of that of η). Assume initial data as in (1.1). Then for any convex function $\varphi: \mathbb{R} \to \mathbb{R}$,

$$E\varphi(\hat{\mathfrak{D}}_R(t;\xi)) \leqslant E\varphi(\hat{\mathfrak{D}}_R(t;\eta)) \tag{2.1}$$

for all t, R.

As immediate consequences, one has the following two corollaries. Here, we set

$$\tilde{\mathfrak{D}}_R(t;\cdot) = \mathfrak{D}_R(t;\cdot) - \mathfrak{D}_R(0;\cdot).$$

Corollary 1. For all t, R,

$$E|\mathfrak{D}_R(t;\xi)| \leqslant E|\mathfrak{D}_R(t;\eta)|. \tag{2.2}$$

Corollary 2. For all t, R.

$$E|\tilde{\mathfrak{D}}_R(t;\xi)| \leqslant E|\tilde{\mathfrak{D}}_R(t;\eta)|. \tag{2.3}$$

Proof of Lemma 2.1. A particle in ξ evolves identically to the corresponding particle in η up until the time it meets a particle of the opposite type; it then disappears. The corresponding particle in η and the particle of opposite type evolve after this time by executing random walks which are (except for their starting positions) independent of $\mathscr{F}_t^\xi$ for all t. Let

$$Y(t) = \hat{\mathfrak{D}}_R(t;\eta) - \hat{\mathfrak{D}}_R(t;\xi) = \mathfrak{D}_R(t;\eta) - \mathfrak{D}_R(t;\xi). \tag{2.4}$$

$Y(t)$ compares the imbalance in the number of A and B particles in D_R for η_t versus ξ_t. Note that since A and B particles evolve in the same manner, $E[Y(t)\mid\mathscr{F}_t^\xi]$ and $E[-Y(t)\mid\mathscr{F}_t^\xi]$ have the same distribution. Consequently,

$$\begin{aligned}
&E[\varphi(\hat{\mathfrak{D}}_R(t;\eta))\mid\mathscr{F}_t^\xi] \\
&\quad = E[\varphi(\hat{\mathfrak{D}}_R(t;\xi) + Y(t))\mid\mathscr{F}_t^\xi] \\
&\quad = \sum_{k=1}^{\infty} P[Y(t)=k\mid\mathscr{F}_t^\xi]\, E[\varphi(\hat{\mathfrak{D}}_R(t;\xi)+k) + \varphi(\hat{\mathfrak{D}}_R(t;\xi)-k)\mid\mathscr{F}_t^\xi] \\
&\qquad + P[Y(t)=0\mid\mathscr{F}_t^\xi]\, E[\varphi(\hat{\mathfrak{D}}_R(t;\xi))\mid\mathscr{F}_t^\xi].
\end{aligned} \tag{2.5}$$

By the convexity of φ, this is

$$\geqslant E[\varphi(\hat{\mathfrak{D}}_R(t;\xi))\mid\mathscr{F}_t^\xi]. \tag{2.6}$$

Taking expectations, we obtain

$$E[\varphi(\hat{\mathfrak{D}}_R(t;\eta))] \geqslant E[\varphi(\hat{\mathfrak{D}}_R(t;\xi))]. \quad \blacksquare$$

Note that the initial data are used only to ensure that the processes, and hence $Y(t)$, are well defined.

In Lemma 2.2 we compute upper bounds for $E[|\mathfrak{D}_R(t;\eta)|]$ and $E[|\hat{\mathfrak{D}}_R(t;\eta)|]$. For both we use

$$\mathscr{M}_R(t;\eta) = \text{total number of particles at time } t \text{ in } D_R. \tag{2.7}$$

Since the subprocesses of η consisting of just the A-type particles and just the B-type particles are Poisson distributed with densities $r_A = r_B$ for all t, one can obtain a copy of η by first designating a particle and then choosing its type with equal probability. $|\mathfrak{D}_R(t;\eta)|$ should therefore be of order $\sqrt{\mathscr{M}_R(t;\eta)}$. For $|\hat{\mathfrak{D}}_R(t;\eta)|$ we apply the same reasoning while noting that the number of particles to pass through ∂D_R (the boundary of D_R) by time t is of order $(\sqrt{t} \wedge R)R^{d-1}$, $2dR^{d-1}$ being the surface area of D_R $((s_1 \wedge s_2) \equiv \min(s_1, s_2))$.

Lemma 2.2. Assume initial data as in (1.1) with $r_A = r_B$. Then

(a) $E|\mathfrak{D}_R(t;\eta)| \leqslant C_1 \sqrt{r_A}\, R^{d/2}$,

(b) $E|\tilde{\mathfrak{D}}_R(t;\eta)| \leqslant C_2 \sqrt{r_A}\, (t^{1/4} \wedge R^{1/2}) R^{(d-1)/2}$.

Proof. The proof of (a) is simple. As mentioned above, $\mathfrak{D}_R(t;\eta)$ is just the sum of $\mathcal{M}_R(t;\eta)$ independent copies of Y, where $P[Y = \pm 1] = 1/2$. Since $E\mathcal{M}_R(t;\eta) = 2r_A R^d$,

$$\sigma^2(\mathfrak{D}_R(t;\eta)) = 2r_A R^d. \tag{2.8}$$

(As noted earlier, we do not worry about whether the cube D_R contains exactly R^d sites.) Part (a) then follows from Jensen's inequality.

Note that application of (a) at times t and 0 implies

$$E|\tilde{\mathfrak{D}}_R(t;\eta)| \leqslant 2C_1 \sqrt{r_A}\, R^{d/2}.$$

To obtain (b), it therefore suffices to show that

$$E|\tilde{\mathfrak{D}}_R(t;\eta)| \leqslant C_2 \sqrt{r_A}\, t^{1/4} R^{(d-1)/2}. \tag{2.9}$$

Let

$$\mathcal{M}_R^{\text{op}}(t;\eta) = \text{total number of particles which are on opposite sides of } \partial D_R \text{ at times 0 and } t. \tag{2.10}$$

Specify ∂D_R so that $\partial D_R \cap \mathbb{Z}^d = \varnothing$ to avoid ambiguity. $\mathcal{M}_R^{\text{op}}$ is then the number of particles which have crossed ∂D_R an odd number of times. For (2.9), it suffices to show

$$E\mathcal{M}_R^{\text{op}}(t;\eta) \leqslant C_3 r_A t^{1/2} R^{d-1}, \tag{2.11}$$

since one can compute variances as in the first part.

By the symmetry of the random walk, the expected number of particles exiting D_R is the same as the expected number entering. To exit, a particle needs to cross one of the $2d$ faces of D_R. Let X_t denote a one-dimensional simple random walk with $X_0 = 0$. By the translation invariance of the motion of the particles, we have

$$\begin{aligned} E\mathcal{M}_R^{\text{op}}(t;\eta) &\leqslant (2r_A)(2d) R^{d-1} \sum_{k=1}^{[R+1]} P[X_t \geqslant k] \\ &\leqslant 4dr_A R^{d-1} \sum_{k=1}^{\infty} P[X_t \geqslant k]. \end{aligned} \tag{2.12}$$

Now, note that

$$E[e^{\theta X_t}] = \exp\{t[(e^{\theta} + e^{-\theta})/2 - 1]\}.$$

Chebyshev's inequality therefore gives

$$P[X_t \geq k] \leq \exp\left\{t\left(\frac{e^{\theta}+e^{-\theta}}{2}-1\right)-\theta k\right\}$$

for $\theta \geq 0$. Plugging in $\theta = \log(1+k/t)$, this

$$= \exp\left\{\frac{k}{2}\left[1-\frac{t}{t+k}+2\log\left(\frac{t}{t+k}\right)\right]\right\}$$

$$\leq e^{-k^2/2(t+k)}.$$

This bound gives

$$\sum_{k=1}^{\infty} P[X_t \geq k] \leq \sum_{k=1}^{\infty} e^{-k^2/2(t+k)} \leq \sum_{k=1}^{\infty} (e^{-k^2/4t}+e^{-k/4})$$

$$\leq \int_0^{\infty} (e^{-x^2/4t}+e^{-x/4})\,dx = \sqrt{\pi t}+4. \tag{2.13}$$

For $t \leq 1$, it is easy to check that $\sum_{k=1}^{\infty} P[X_t \geq k] \leq \text{const. } t$. Together with (2.12) and (2.13), this implies (2.11), and hence demonstrates (b). ∎

We also observe that:

Lemma 2.3. Assume initial data as in (1.1) with $r_A = r_B$. Then for large R and appropriate $C_4 > 0$,

$$E|\mathfrak{D}_R(0;\xi)| \geq C_4\sqrt{r_A}\,R^{d/2}.$$

Proof. As in Lemma 2.2, $\mathfrak{D}_R(0;\xi) = \mathfrak{D}_R(0;\eta)$ is the sum of $\mathscr{M}_R(0;\xi)$ independent copies of Y, with $P[Y=\pm 1]=1/2$. Since $E\mathscr{M}_R(0;\xi) = 2r_A R^d$ and $\sigma^2(\mathscr{M}_R(0;\xi)) = 2r_A R^d$, by Chebyshev's inequality,

$$P[\mathscr{M}_R(0;\xi) \geq r_A R^d] \geq 1 - 2/r_A R^d \geq 1/2 \tag{2.14}$$

for large enough R. By the central limit theorem,

$$P[|\mathfrak{D}_R(0;\xi)| \geq \sqrt{r_A}\,R^{d/2} \mid \mathscr{M}_R(0;\xi) \geq r_A R^d] \geq C_5 \tag{2.15}$$

for appropriate $C_5 > 0$. So

$$E|\mathfrak{D}_R(0;\xi)| \geq C_5\sqrt{r_A}\,R^{d/2}/2 = C_4\sqrt{r_A}\,R^{d/2}. \quad ∎$$

We can now follow the outline presented in the introduction through (1.15) to prove Proposition 1. Proposition 1 supplies the lower bound in Theorem 1 for $d<4$ and part of the lower bound for $d=4$.

Proposition 1. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$ according to (1.1) with $r_A = r_B > 0$. Then for $t \geq 1$,

$$\rho_A(t) = \rho_B(t) \geq c\sqrt{r_A}/t^{d/4} \tag{2.16}$$

for appropriate $c > 0$.

Proof. By Lemma 2.2(b),

$$E|\tilde{\mathfrak{D}}_R(t;\eta)| \leq C_2\sqrt{r_A}\, t^{1/4} R^{(d-1)/2}.$$

So by Corollary 2 of Lemma 2.1,

$$E|\tilde{\mathfrak{D}}_R(t;\xi)| \leq C_2\sqrt{r_A}\, t^{1/4} R^{(d-1)/2}.$$

On the other hand, by Lemma 2.3,

$$E|\mathfrak{D}_R(0;\xi)| \geq C_4\sqrt{r_A}\, R^{d/2}.$$

Consequently,

$$\begin{aligned} E|\mathfrak{D}_R(t;\xi)| &\geq E|\mathfrak{D}_R(0;\xi)| - E|\mathfrak{D}_R(t;\xi) - \mathfrak{D}_R(0;\xi)| \\ &\geq C_4\sqrt{r_A}\, R^{d/2} - C_2\sqrt{r_A}\, t^{1/4} R^{(d-1)/2}. \end{aligned} \tag{2.17}$$

Now set $R = a\sqrt{t}$. For a large enough, (2.17) is at least $b\sqrt{r_A}\, R_t^{d/2}$ for some $b > 0$. By symmetry,

$$\rho_A(t) \geq \tfrac{1}{2} R_t^{-d} E[|\mathfrak{D}_R(t;\xi)|].$$

Plugging in the bound for $E[\mathfrak{D}_R(t;\xi)]$ and substituting for R, we obtain

$$\rho_A(t) \geq c\sqrt{r_A}/t^{d/4},$$

with $c = b/2a^{d/2}$. ▮

3. *Lower Bounds for Equal Densities, $d \geq 4$*

In this section we will show that $\rho_A(t)$ decays at most like c/t, $c > 0$, for all dimensions d if $r_A = r_B$. For $d \geq 4$, this will provide us with the correct lower bound. (As was shown in the previous section, for $d < 4$ this bound is not sharp.) The bound c/t is intuitive—as explained in the introduction, it corresponds to the mean field approximation which assumes that the positions of A and B particles are independent. The bound even goes in the "right" direction: the positions of the opposite types of particles should actually not be completely independent, but somewhat negatively correlated. Therefore, these particles should be somewhat less likely to collide, and the density should fall less quickly than c/t. One can also come up with other heuristics. For instance, the densities $\rho_A(t)$, $\rho_B(t)$ should decay less rapidly than for the annihilating random walk given in the introduction. Here, annihilation does not always occur when particles meet (when they are of the same type), which should slow things down. Since the density of the annihilating random walk dies off at most like c/t, so should the density of our two-particle system.

Unfortunately, we were unable to give a proof along these lines. Our indirect proof here makes use of several lemmas. Lemma 3.1 compares two copies of our two-particle annihilating random walk which share the same percolation substructure and whose initial states differ only by a single particle at y; the lemma says this difference remains for all time, with the extra particle moving as a random walk X_t^y. As a consequence, at any time along this path, one or the other of these processes must have at least one particle. Roughly speaking, this says that along this path, the probability is $\sim 1/2$ that the process will have a particle. This is formulated in Lemma 3.4.

Consider a random set $\mathscr{A}$ in $\mathbb{Z}^d$ consisting of the occupied sites of a Poisson random measure with mean $m = 1/Mt$. One can assume that $\mathscr{A} \subset \{x: \xi_0 \neq 0\}$. View the process ξ_s for $0 \leqslant s \leqslant t$, paying special attention to the evolution of "associated" random walks, one starting at each point of $\mathscr{A}$. One can show, as in Lemma 3.3, that the probability of these random walks intersecting by time t is not too large (say 1/2) if M is large. (One actually needs to show a bit more in the lemma.) So the density of these random walks (which are allowed to coalesce with each other) is at least $1/2Mt$ at time t. On the other hand, if these random walks correspond to paths constructed from ξ in the above heuristics for Lemmas 3.1 and 3.4, then the probability that such a path is occupied by at least one A or B particle at time t is $\sim 1/2$. This should imply that the density of occupied sites at time t, $\rho_A(t) + \rho_B(t)$, is at least of order $1/4Mt$, which is the type of result we want. Proposition 2 makes this precise.

We need some notation for Lemma 3.1. For $x \in \mathbb{Z}^d$, set

$$\begin{aligned} \Lambda_t(x) &= \xi_{t-}(x) && \text{if} \quad \xi_{t-}(x) > 0, \\ &= \xi_{t-}(x) - 1 && \text{if} \quad \xi_{t-}(x) \leqslant 0. \end{aligned} \tag{3.1}$$

Note that $\Lambda_t(x)$ is left continuous. Let X_t^y, $y \in \mathbb{Z}^d$, be the random walk starting at y which moves according to level $\Lambda_t(X_{t-}^y)$ of the percolation substructure $\mathscr{P}$ of ξ_t given in the introduction. That is, if an alarm clock goes off at level $\Lambda_t(X_{t-}^y)$, then X_t^y moves according to the corresponding arrow. Random walks $X_t^{y_1}$, $X_t^{y_2}$ with $X_t^{y_1} \neq X_t^{y_2}$ move independently; their movement is also independent of ξ_{0-}. Also, introduce the processes ξ_t^y, $y \in \mathbb{Z}^d$, where ξ_t^y evolves as does ξ_t according to $\mathscr{P}$, but with initial measure

$$\begin{aligned} \xi_0^y(x) &= \xi_0(x) - 1 \qquad \text{for} \quad x = y, \\ &= \xi_0(x) \qquad \text{for} \quad x \neq y. \end{aligned} \tag{3.2}$$

Lemma 3.1. For ξ_t^y as defined above,

$$\begin{aligned} \xi_t^y(x) &= \xi_t(x) - 1 \qquad \text{for} \quad x = X_t^y, \\ &= \xi_t(x) \qquad \text{for} \quad x \neq X_t^y. \end{aligned} \tag{3.3}$$

Proof. Assume that (3.3) holds for $t < T$ for some (possibly random) T. (3.3) can only be violated at $t = T$ if at time T there is an arrow in $\mathscr{P}$ from X_{T-}^y. If the arrow occurs on level $A_T(X_{T-}^y)$, then $X_T^y \neq X_{T-}^y$. Moreover, if $\xi_{T-}(X_{T-}^y) > 0$, then the extra B particle in ξ at X_{T-}^y moves to X_T^y, whereas if $\xi_{T-}(X_{T-}^y) \leq 0$, then the extra A particle in ξ^y at X_{T-}^y moves to X_T^y. (This is the motivation for (3.1).) In either case, (3.3) will continue to hold at $t = T$. If the arrow occurs on a level other than $A_T(X_{T-}^y)$, then the same type of particle (type A, type B, or no particle), moves for both ξ and ξ^y. Here $X_T^y = X_{T-}^y$, and so (3.3) continues to hold at $t = T$. Therefore, (3.3) cannot be violated at $t = T$; induction shows that (3.3) holds for all t. ∎

We are also interested in the following consequence of Lemma 3.1.

Corollary 1. For ξ_t^y defined as above,

$$\text{either} \quad \xi_t(X_t^y) \neq 0 \quad \text{or} \quad \xi_t^y(X_t^y) \neq 0 \tag{3.4}$$

for each t. That is, at least one of the two processes ξ_t, ξ_t^y has a particle at X_t^y.

In Section 6, we will also use random walks X_t^y to compare processes ξ and $\tilde{\xi}$ sharing the same percolation substructure but beginning from different initial states. There, the initial states will be unequal at many sites, with the difference possibly being greater than one. For $\tilde{\xi}$, type A particles will die upon reaching certain boundaries; this will correspond to the birth of random walks. Despite these embellishments, the difference between ξ and $\tilde{\xi}$ will be governed by random walks in a manner analogous to (3.3).

Before proceeding with additional preparation for Proposition 2, we pause to make a simple observation concerning two-particle annihilating random walks ξ and ξ' with $\xi_0 \leq \xi_0'$ (i.e., $\xi_0(x) \leq \xi_0'(x)$ for all x). Lemma 3.2 is a maximum principle; it will be referred to in Sections 6 and 7. The proof is similar to that of Lemma 3.1. Here, ${}^R\xi_t$ denotes the process which is restricted initially to D_R, that is,

$${}^R\xi_0 = \xi_0 \cap D_R.$$

Lemma 3.2. Suppose ξ_t and ξ'_t have the same percolation substructure with $\xi_0 \leqslant \xi'_0$. Then

$$\xi_t \leqslant \xi'_t \qquad \text{for all} \quad t. \tag{3.5}$$

Proof. It suffices to show that for given R,

$${}^R\xi_t(x) \leqslant {}^R\xi'_t(x) \qquad \text{for all} \quad t, x. \tag{3.6}$$

Since

$$\lim_{R\to\infty} {}^R\xi_t(x) = \xi_t(x), \qquad \lim_{R\to\infty} {}^R\xi'_t(x) = \xi'(x),$$

(3.5) follows from (3.6). Suppose now that (3.6) holds for $t < T$ for some (possibly random) T. At given x, ${}^R\xi_t$ (or ${}^R\xi'_t$) only changes at $t = T$ if either (i) there is an arrow from x, or (ii) there is an arrow to x. It is easy to check that under (3.6) for $t < T$ and (i),

$${}^R\xi_T(x) \leqslant {}^R\xi'_T(x). \tag{3.7}$$

Suppose that (ii) holds and denote by ${}^R\varphi_T(x) = -1,\ 0,\ 1$ the type of particle arriving at x (A-particle $= -1$, B-particle $= 1$, no particle $= 0$). Then,

$${}^R\xi'_T(x) - {}^R\xi_T(x) = {}^R\xi'_{T-}(x) - {}^R\xi_{T-}(x) + {}^R\varphi'_T(x) - {}^R\varphi_T(x). \tag{3.8}$$

By our assumption (3.6) for $t < T$,

$${}^R\varphi_T(x) \leqslant {}^R\varphi'_T(x).$$

So from (3.8),

$${}^R\xi'_T(x) - {}^R\xi_T(x) \geqslant {}^R\xi'_{T-}(x) - {}^R\xi_{T-}(x) \geqslant 0 \tag{3.9}$$

in case (ii) as well. (3.7) and (3.9), together with an induction argument, show that (3.6) holds for all t, x. ∎

We will require additional notation for Lemmas 3.3 and 3.4. Let $\zeta^k(x)$, $x \in \mathbb{Z}^d$, $k = 1, 2$, be random variables taking values 0, 1 at each site with $P[\zeta^k(x) = 1] = m$ for some $0 < m < 1$. We assume (ζ^1, ζ^2) is chosen to be independent at different sites (also, independent when conditioned on ξ_{0-}), and that $\mathscr{P}$ is independent of ζ^1, ζ^2 and ξ_{0-}. ($\zeta^k(x)$ will be specified before Lemma 3.4.) Set

$$\mathscr{A}_m^k = \{x : \zeta^k(x) = 1\}. \tag{3.10}$$

For given m, $\mathscr{A}_m^1$ and $\mathscr{A}_m^2$ should be thought of as low density random sets having independent coordinates and which are constructed independently of $\mathscr{P}$. Introduce the processes $\tilde{\xi}^y$, $y \in \mathbb{Z}^d$, which evolve as does ξ according to $\mathscr{P}$, but with initial measure

$$\begin{aligned} \tilde{\xi}_0^y(x) &= \xi_0(x) + 1 \qquad \text{for} \quad x = y, \\ &= \xi_0(x) \qquad\qquad \text{for} \quad x \neq y. \end{aligned} \tag{3.11}$$

One can define $\tilde{\Lambda}_t^y(x)$ and $\tilde{X}_t^y$ analogous to $\Lambda_t(x)$ and X_t^y, but relative to $\tilde{\xi}^y$ instead of ξ. Also, introduce

$$\tau^y = \inf\{t : X_t^y = X_t^z,\, z \in \mathscr{A}_m^2 - y, \text{ or } X_t^y = \tilde{X}_t^z,\, z \in \mathscr{A}_m^1 - y\}. \tag{3.12}$$

($\mathscr{A}_m^k - y$ denotes $\mathscr{A}_m^k \cap \{y\}^c$.) Up until time τ^y, the random walk X_t^y does not meet any of the random walks X_t^z, $\tilde{X}_t^z$. Note that τ^y is independent of the event $\{y \in \mathscr{A}_m^k\}$.

Lemma 3.3 says that if the density of random walks is small enough, then the probability they do not hit by a given time is large.

Lemma 3.3. $P[t < \tau^y;\, y \in \mathscr{A}_m^2] \geqslant m[1 + 2e^2(t+1)\log(1-m)]$. In particular, for $m \leqslant 1/32(t+1)$,

$$P[t < \tau^y;\; y \in \mathscr{A}_m^2] \geqslant m/2. \tag{3.13}$$

Proof. Clearly,

$$\begin{aligned} P[\tau^y \leqslant t] \leqslant \sum_z P[X_s^y = X_s^z, \text{ some } s \in (0, t];\, z \in \mathscr{A}_m^2 - y] \\ + \sum_z P[X_s^y = \tilde{X}_s^z, \text{ some } s \in (0, t];\, z \in \mathscr{A}_m^1 - y]. \end{aligned} \tag{3.14}$$

We estimate the first sum; the same reasoning pertains to the second. It will be helpful to introduce the random walks Y_s^z, $z \in \mathbb{Z}^d$, with $Y_0^z = z$, which move independently of each other (even when their positions coincide). The first sum in (3.14) equals

$$\begin{aligned} &\sum_z P[Y_s^y = Y_s^z, \text{ some } s \in [0, t];\, z \in \mathscr{A}_m^2 - y] \\ &\leqslant \sum_{k=1}^{[t]+1} \sum_z P[Y_s^y = Y_s^z, \text{ some } s \in (k-1, k];\, z \in \mathscr{A}_m^2 - y]. \end{aligned} \tag{3.15}$$

The probability that a rate-2 random walk does not move in one unit of time is e^{-2}. So by the strong Markov property, (3.15) is at most

$$e^2 \sum_{k=1}^{[t]+1} \sum_z P[Y_k^y = Y_k^z; z \in \mathscr{A}_m^2 - y] = e^2 \sum_{k=1}^{[t]+1} E[\# z \in \mathscr{A}_m^2 - y: Y_k^y = Y_k^z]. \tag{3.16}$$

The set $\{Y_0^z, z \in \mathscr{A}_m^2 - y\}$ is dominated by a Poisson random measure with mean $-\log(1-m)$; the same is thus true for all k. So (3.16) is at most

$$-e^2(t+1)\log(1-m). \tag{3.17}$$

It follows from (3.15)–(3.17) that the first sum in (3.14) is at most $-e^2(t+1)\log(1-m)$; applying the same reasoning to the second sum gives

$$P[\tau^y \leqslant t] \leqslant -2e^2(t+1)\log(1-m). \tag{3.18}$$

Consequently,

$$\begin{aligned} P[t < \tau^y; y \in \mathscr{A}_m^2] &= P[t < \tau^y]\, P[y \in \mathscr{A}_m^2] \\ &\geqslant m[1 + 2e^2(t+1)\log(1-m)]. \end{aligned}$$

For $m \leqslant 1/32(t+1)$, this is at least $m/2$. ∎

We recall that under (1.1), $\xi_0(x) = \xi^2(x) - \xi^1(x)$, where $\xi^1(x)$ and $\xi^2(x)$ are independent Poisson random variables with means r_A and r_B with $r_A = r_B$. Extend our space $(\Omega, \mathscr{F}, P)$ to include uniform $[0, 1]$ random variables V^x, $x \in \mathbb{Z}^d$, which are independent of everything else. Set

$$\mathscr{A}^k = \{x: \xi^1(x) = [r_B] + k\}, \qquad k = 1, 2.$$

The reason for this definition of $\mathscr{A}^k$ will become clear in Lemma 3.4. ($\mathscr{A}^k$ could be defined in a variety of ways.) Fix $m > 0$ with $m \leqslant P[x \in \mathscr{A}^2]$ ($m \leqslant P[x \in \mathscr{A}^1]$ follows). One can of course choose a_m^k so that for $\mathscr{B}_m^k \equiv \{x: V^x \leqslant a_m^k\}$ and $\mathscr{A}_m^k \equiv \mathscr{A}^k \cap \mathscr{B}_m^k$,

$$P[x \in \mathscr{A}_m^k] = m.$$

Note that $\mathscr{B}_m^k$ serves to "thin out" $\mathscr{A}^k$. It is easy to check that $\mathscr{A}_m^k$ and the corresponding random variables $\zeta^k(x)$ have the properties specified just before (3.10). One can of course choose $m \leqslant 1/32(t+1)$ so (3.13) holds.

We introduce $E_m^y(t)$, where

$$E_m^y(t) = \{\omega: t < \tau^y, y \in \mathscr{A}_m^2\}. \tag{3.19}$$

When convenient, we will drop indices. We also introduce the map $T^y: \Omega \to \Omega$, with $T^y(\omega) = \omega'$ sharing the same percolation substructure $\mathscr{P}$ as ω, but with

$$\begin{aligned} \xi^1(x, \omega') &= \xi^1(x, \omega) && \text{for all } x, \\ \xi^2(x, \omega') &= [\xi^2(x, \omega) - 1] \vee 0 && \text{if } x = y, \\ &= \xi^2(x, \omega) && \text{if } x \neq y, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} V^x(\omega') &= (r_B/([r_B]+2))\, V^x(\omega) && \text{if } x = y, \\ &= V^x(\omega) && \text{if } x \neq y. \end{aligned} \tag{3.21}$$

For $y \in \mathscr{A}^2$, one has

$$\begin{aligned} \xi_0(x, \omega') &= \xi_0(x, \omega) - 1 && \text{if } x = y, \\ &= \xi_0(x, \omega) && \text{if } x \neq y. \end{aligned} \tag{3.22}$$

So under $y \in \mathscr{A}^2$, $\xi_t(x, \omega') = \xi_t^y(x, \omega)$ for all t. The point of (3.21) is the following simple lemma.

Lemma 3.4. Assume initial data as in (1.1). For $G \in \mathscr{F}_\infty$,

$$P[T^y(G \cap E_m^y(t))] = P[G \cap E_m^y(t)].$$

Proof. T^y alters just $\xi^2(y)$ and V^y. Since the initial state at y is independent of $\mathscr{P}$ and the initial state at $x \neq y$,

$$\frac{P[T^y(G \cap E)]}{P[G \cap E]} = \frac{r_B}{[r_B]+2} \, \frac{P[\xi^2(y) = [r_B]+1]}{P[\xi^2(y) = [r_B]+2]}, \tag{3.23}$$

where the first term on the right comes from (3.21), V^y being uniformly distributed. Since $\xi^2(y)$ is Poisson distributed with mean r_B, the second term equals $([r_B]+2)/r_B$. The two terms on the right therefore give 1 when multiplied. ∎

It is also easy to check that

$$T^y(\{y \in \mathscr{A}_m^2\}) = \{y \in \mathscr{A}_m^1\}. \tag{3.24}$$

For $y \in \mathscr{A}_m^1$ (so $\xi_2(y, \omega') = \xi^2(y, \omega) - 1$),

$$\tilde{X}_t^y(\omega') = X_t^y(\omega). \tag{3.25}$$

The corollary to Lemma 3.1 can therefore be reworded as saying that

$$\xi_t(X_t^y(\omega), \omega) = 0 \Rightarrow \xi_t(\tilde{X}_t^y(\omega'), \omega') \neq 0 \tag{3.26}$$

for all t and $y \in \mathscr{A}_m^1$. It also follows from Lemma 3.1 that for $x \neq y$,

$$\tilde{X}_t^x(\omega') = \tilde{X}_t^x(\omega) \tag{3.27}$$

if $\tilde{X}_s^x(\omega) \neq X_s^y(\omega)$ for all $s \in [0, t]$ (since then $\xi_s(\tilde{X}_s^x(\omega), \omega) = \xi_s(\tilde{X}_s^x(\omega), \omega')$).

We combine Lemmas 3.1, 3.3, 3.4 to show Proposition 2, which gives a lower bound on the limiting behavior of $\rho_A(t)$ under equal initial densities. The result complements Proposition 1, and supplies the lower bound in Theorem 1 for $d > 4$ and part of the lower bound for $d = 4$.

Proposition 2. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$ according to (1.1) with $r_A = r_B > 0$. Then for large enough t,

$$\rho_A(t) = \rho_B(t) \geqslant c/t, \tag{3.28}$$

where $c > 0$ does not depend on r_A or d.

Proof. For any set $G \in \mathscr{F}_\infty$,

$$\begin{aligned} P[E] &= P[E \cap G] + P[E \cap G^c] \\ &= P[T^0(E \cap G)] + P[E \cap G^c] \end{aligned}$$

according to Lemma 3.4, with $y = 0$. Let $G = \{\omega : \xi_t(X_t^0) = 0\}$. Also, set $m = (1/32(t+1)) \wedge P[0 \in \mathscr{A}^2]$. By Lemma 3.3,

$$P[E] \geqslant 1/64(t+1)$$

for large enough t. So either

$$P[E \cap G^c] \geqslant 1/128(t+1) \qquad \text{or} \qquad P[T^0(E \cap G)] \geqslant 1/128(t+1). \tag{3.29}$$

(3.28) will follow once we show that

$$P[\omega : \xi_t(0) \neq 0] \geqslant P[E \cap G^c] \tag{3.30a}$$

and

$$P[\omega : \xi_t(0) \neq 0] \geqslant P[T^0(E \cap G)]. \tag{3.30b}$$

The reasoning for (3.30a) is as follows. Of course,

$$\begin{aligned} &P[\omega : \xi_t(0) \neq 0)] \\ &\quad \geqslant P[\exists !\, y \in \mathscr{A}_m^2 : X_t^y = 0;\ \xi_t(0) \neq 0] \\ &\quad = \sum_{y \in \mathbb{Z}^d} P[y \in \mathscr{A}_m^2;\ X_t^y \neq X_t^z,\ z \in \mathscr{A}_m^2 - y;\ X_t^y = 0,\ \xi_t(0) \neq 0] \end{aligned} \tag{3.31}$$

by restricting the set. Note that the uniqueness condition "$\exists !$" is needed to obtain this decomposition. The process ξ is translation invariant, and so the last quantity in (3.31) equals

$$\sum_{y\in\mathbb{Z}^d} P[0\in\mathscr{A}_m^2;\, X_t^0\neq X_t^z,\, z\in\mathscr{A}_m^2-0;\, X_t^0=-y;\, \xi_t(-y)\neq 0]$$
$$=P[0\in\mathscr{A}_m^2;\, X_t^0\neq X_t^z,\, z\in\mathscr{A}_m^2-0;\, \xi_t(X_t^0)\neq 0]. \tag{3.32}$$

But it is easy to see that

$$\begin{aligned} &P[0\in\mathscr{A}_m^2;\, X_t^0\neq X_t^z,\, z\in\mathscr{A}_m^2-0;\, \xi_t(X_t^0)\neq 0]\\ &\geqslant P[0\in\mathscr{A}_m^2;\, t<\tau^0;\, \xi_t(X_t^0)\neq 0]\\ &=P[E\cap G^c]. \end{aligned} \tag{3.33}$$

(3.31)–(3.33) imply (3.30a).

The reasoning for (3.30b) follows in much the same manner. We have

$$P[\omega':\xi_t(0)\neq 0]\geqslant P[\exists !\, y\in\mathscr{A}_m^1:\tilde{X}_t^y=0;\, \xi_t(0)\neq 0] \tag{3.34}$$

by restricting the set. As in (3.31)–(3.32), this equals

$$P[0\in\mathscr{A}_m^1;\, \tilde{X}_t^0\neq\tilde{X}_t^z,\, z\in\mathscr{A}_m^1-0;\, \xi_t(\tilde{X}_t^0)\neq 0].$$

By inclusion, this is at least

$$P[0\in\mathscr{A}_m^1;\, \tilde{X}_s^0\neq\tilde{X}_s^z,\, z\in\mathscr{A}_m^1-0,\, s\in[0,t];\, \xi_t(\tilde{X}_t^0)=0\}]. \tag{3.35}$$

On account of (3.24)–(3.27), (3.35) is at least

$$P[T^0\{0\in\mathscr{A}_m^2;\, X_s^0\neq\tilde{X}_s^z,\, z\in\mathscr{A}_m^1-0,\, s\in[0,t];\, \xi_t(X_t^0)=0\}]. \tag{3.36}$$

(To see this, plug in the corresponding terms. The substitution of $T^0\{\xi_t(X_t^0)=0\}$ for $\{\xi_t(\tilde{X}_t^0)\neq 0\}$ uses (3.26) and thus the corollary of Lemma 3.1; it is responsible for the inequality.) This is at least

$$P[T^0\{0\in\mathscr{A}_m^2;\, t<\tau^0;\, \xi_t(X_t^0)=0\}]=P[T^0(E\cap G)]. \tag{3.37}$$

(3.34)–(3.37) imply (3.30b). This completes the proof. ∎

4. Upper Bounds for Equal Densities

In this section we will show that if $r_A=r_B$, then $\rho_A(t)$ decays at least like $C\sqrt{r_A}/t^{d/4}$ for $d<4$, C/t for $d>4$, and $C(\sqrt{r_A}\vee 1)/t$ for $d=4$. This result, Proposition 3, together with the lower bounds given by Propositions 1 and 2 of the previous two sections, demonstrates Theorem 1. To prepare for Proposition 3, we present a series of six lemmas. Lemmas 4.1 and 4.2 are elementary. Lemma 4.1 states that pairs of particles starting a given distance apart hit at least with a certain probability by an appropriate elapsed time. By Lemma 4.2, the total number of particles present will therefore decay at least at a certain rate if both types of particles are present in the same cube D_R in large enough numbers. For this procedure to be efficient, we will need $R \leqslant \sqrt{t}$ in general. So although there will be plenty of particles of both types in $D_{\sqrt{t}}$, we still need a stirring mechanism to distribute the particles evenly in smaller regions. This is given in Lemmas 4.3–4.5. (Lemma 4.3 is a simple large deviations estimate which will also be used later.) Lemma 4.6 then uses the previous lemmas to show that the number of particles will continue to decay rapidly as long as there are substantial numbers of both types of particles in $D_{\sqrt{t}}$. But from the first two lemmas of Section 2, $|\mathfrak{D}_{\sqrt{t}}(t;\xi)|$, the net difference in the numbers of the two types of particles, is comparatively small. The number of particles will therefore continue to decay until the total number of particles left in $D_{\sqrt{t}}$ is small; this is the conclusion in Proposition 3.

For Lemma 4.1, we introduce the following notation. Let ${}^2Y_s^x$ denote a rate-2 random walk starting at x, and set

$$\tau = \inf\{s\colon {}^2Y_s^x = 0\}. \tag{4.1}$$

The norm $\|x\|$ is chosen so $x \in D_R$ iff $\|x\| \leqslant R$. We are interested in obtaining lower bounds for

$$H_s(x) = P[\tau < s]. \tag{4.2}$$

This can be conveniently expressed in terms of the Green's function

$$G_t(x) = \int_0^t P[{}^2Y_s^x = 0]\, ds. \tag{4.3}$$

The lemma is from ref. 12.

Lemma 4.1. If $x \in \mathbb{Z}^d$ with $\|x\| = R$, then for appropriate $c_1 > 0$ (depending on d),

$$\begin{aligned} H_{R^2}(x) &\geqslant c_1, & d&=1,\\ &\geqslant c_1/\log R, & d&=2,\\ &\geqslant c_1 R^{2-d}, & d&\geqslant 3. \end{aligned} \tag{4.4}$$

Proof. $d=1$ follows from the central limit theorem. $d \geqslant 2$ (and $d=1$, if desired) follows from the inequality

$$H_t(x) \geqslant G_t(x)/G_t(0)$$

together with the following well-known asymptotics for G: as $R = \|x\| \to \infty$,

$$\begin{aligned} G_{R^2}(x) &\approx \alpha_2 && d=2,\\ &\approx \alpha_d R^{2-d} && d\geqslant 3;\\ G_{R^2}(0) &\approx \beta_2 \log R && d=2,\\ &\approx \beta_d && d\geqslant 3. \end{aligned}$$

The local central limit theorem gives these asymptotics. (See Spitzer[14]). ∎

We will use the following notation. Let $\mathfrak{D}_R^A(t)$ ($\mathfrak{D}_R^B(t)$) denote the number of A particles (B particles) in the cube D_R for the process ξ_t. Set

$$\begin{aligned} \mathfrak{D}_R^m(t) &= \mathfrak{D}_R^A(t) \wedge \mathfrak{D}_R^B(t),\\ \mathfrak{D}_R^M(t) &= \mathfrak{D}_R^A(t) \vee \mathfrak{D}_R^B(t), \\ \mathfrak{D}_R^T(t) &= \mathfrak{D}_R^A(t) + \mathfrak{D}_R^B(t) = \mathfrak{D}_R^m(t) + \mathfrak{D}_R^M(t). \end{aligned} \tag{4.5}$$

Recall that

$$\mathfrak{D}_R(t) = \mathfrak{D}_R^B(t) - \mathfrak{D}_R^A(t).$$

Also, set

$$h_d(R) = \min\{H_{R^2}(x)\colon \|x\| \leqslant R\}. \tag{4.6}$$

Lemma 4.2. Assume that ξ has translation invariant initial data. Then

$$E[\mathfrak{D}_R^T(0)] - E[\mathfrak{D}_R^T(R^2)] \geqslant h_d(R)\, E[\mathfrak{D}_R^m(0)]. \tag{4.7}$$

Proof. Enumerate by $x_1, x_2, \ldots, x_n$ the positions of the particles of minority type at time 0 in D_R, and by $z_1, z_2, \ldots, z_N$, $N \geqslant n$, the positions of particles of majority type. Let $X_s^{x_k}$ and $Z_s^{z_k}$ denote the random walks executed by these particles for the process η without interactions. Until the time τ_k^m (τ_k^M) at which the particle starting at x_k (z_k) in ξ is annihilated, the particle moves according to X^{x_k} (Z^{z_k}). No matter what our choice of x_k and z_k,

$$P[X_s^{x_k} = Z_s^{z_k}, \text{ some } s \leqslant R^2 \mid \mathscr{F}_0] \geqslant h_d(R). \tag{4.8}$$

(Recall that $\mathscr{F}_0$ gives the initial configuration.) Denote by σ_k the time at which these two random walks first meet.

The bound (4.7) is obtained by pairing up the particles starting at x_k

and z_k, $k \leqslant n$. If $\sigma_k = \tau_k^m = \tau_k^M$, then both particles disappear at time σ_k. Otherwise, at least one of the particles has already disappeared by σ_k. So by (4.8), the probability that one or the other of these particles has disappeared by time R^2 is at least $h_d(R)$. Therefore,

$$E[\# k \leqslant n: \tau_k^m \wedge \tau_k^M \leqslant R^2] \geqslant h_d(R)\, E[\mathfrak{D}_R^m(0)]. \tag{4.9}$$

Since we are assuming ξ_0 is translation invariant, this implies (4.7). ∎

Lemma 4.3 is a simple but useful large deviations estimate.

Lemma 4.3. Let $X_1,..., X_n$ be independent random variables with $P[X_k = 1] = p_k$, $P[X_k = 0] = 1 - p_k$, and $\sum_{k=1}^n p_k = m$. Set $S_n = \sum_{k=1}^n X_k$. For appropriate $\beta > 0$ (independent of p_k),

$$P[S_n - m \leqslant -\delta m],\ P[S_n - m \geqslant \delta m] \leqslant e^{-\beta\delta(\delta \wedge 1)m} \tag{4.10}$$

for each $\delta > 0$.

Proof. $E[e^{\theta(X_k - p_k)}] = (1 - p_k)e^{-\theta p_k} + p_k e^{\theta(1 - p_k)}$.

So,

$$E[e^{\theta(S_n - m)}] = \prod_{k=1}^{n} [(1 - p_k)e^{-\theta p_k} + p_k e^{\theta(1 - p_k)}].$$

By Chebyshev's inequality for $\theta > 0$,

$$P[S_n - m \geqslant \delta m] \leqslant e^{-\delta\theta m} \prod_{k=1}^{n} [(1 - p_k)e^{-\theta p_k} + p_k e^{\theta(1 - p_k)}]. \tag{4.11}$$

For θ small, simple estimation shows that the quantities in brackets at the right are at most $1 + \theta^2 p_k$. So, the sum of their logarithms is at most

$$\theta^2 \sum_{k=1}^{n} p_k = \theta^2 m.$$

The left side of (4.11) is therefore at most $\exp\{-\theta m(\delta - \theta)\}$. For $\theta = (\delta \wedge \theta_o)/2$, appropriate $0 < \theta_o \leqslant 1$, we therefore get

$$P[S_n - m \geqslant \delta m] \leqslant e^{-\delta(\delta \wedge \theta_o)m/4} \leqslant e^{-\beta\delta(\delta \wedge 1)m},$$

where $\beta = \theta_o/4$. By considering instead $e^{-\theta(X_k - p_k)}$, one can reason as above to get

$$P[S_n - m \leqslant -\delta m] \leqslant e^{-\beta\delta(\delta \wedge 1)m}. \quad ∎$$

Corollary 1. Let $X_1,\ldots, X_n$ be i.i.d. random variables with $P[X_1=1]=p$, $P[X_1=0]=1-p$. Set $S_n=\sum_{k=1}^n X_k$. Then for $\beta>0$ as above,

$$P[S_n \leqslant (1-\delta)np],\ P[S_n \geqslant (1+\delta)np] \leqslant e^{-\beta\delta(\delta \wedge 1)np}. \tag{4.12}$$

In Lemma 4.4, we will consider cubes D_R and $D_{r,j}$, $j \in J$, where r divides R and $\{D_{r,j}, j \in J\}$ partitions D_R into smaller cubes; r is to be thought of as being much smaller than R. Set $q=(r/R)^d$. The lemma says that if $\mathfrak{D}_R^m(0) \geqslant L$, then one can give the lower bounds $\mathfrak{D}_{r,j}^m(s;\eta) \geqslant \beta_1 qL$ for appropriate s. (A little thought shows that one cannot expect more.) Recall that η is the system corresponding to ξ whose particles do not interact.

Lemma 4.4. Suppose that $\mathfrak{D}_R^m(0) \geqslant L$. Then for appropriate $\beta_1>0$ (not depending on r or R),

$$P[\mathfrak{D}_{r,j}^m(s;\eta) \geqslant \beta_1 qL] \geqslant 1-e^{-\beta_1 qL} \tag{4.13}$$

for all $s \in [R^2/2, R^2]$ and all $j \in J$.

Proof. It is not difficult to show by using the local central limit theorem that for the random walk Y_s^x, and all x, $y \in D_R$ and $s \in [R^2/2, R^2]$,

$$P[Y_s^x = y] \geqslant \beta_2/R^d$$

for appropriate $\beta_2>0$. Consequently,

$$P[Y_s^x \in D_{r,j}] \geqslant \beta_2 q \tag{4.14}$$

for all $j \in J$. (As we have warned in the beginning of Section 2, we are cheating here in retaining the constant β_2 by pretending that all cubes $D_{r,j}$ contain the same number of lattice points.) Introduce i.i.d. Bernoulli random variables $X_1,\ldots, X_L$ with $P[X_1=1]=\beta_2 q$ and $P[X_1=0]=1-\beta_2 q$. Let $S_L=\sum_{k=1}^L X_k$. On account of (4.14),

$$P[\mathfrak{D}_{r,j}^A(s;\eta) \leqslant \beta_2 qL/2] \leqslant P[S_L \leqslant \beta_2 qL/2], \tag{4.15}$$

since there are at least L type-A particles starting in D_R. By Lemma 4.3, this is at most $e^{-\beta_1 qL}/2$ for appropriate $\beta_1>0$. Similarly,

$$P[\mathfrak{D}_{r,j}^B(s;\eta) \leqslant \beta_2 qL/2] \leqslant e^{-\beta_1 qL}/2. \tag{4.16}$$

For $\beta_1 \leqslant \beta_2/2$, these bounds imply (4.13). ∎

Corollary 1. Suppose that $E[\mathfrak{D}_R^m(0)] \geqslant L_1$, where $L_1 \geqslant c_2/q$ for appropriate $c_2>0$ (not depending on r or R). Then

$$E[\mathfrak{D}_{r,j}^m(s;\eta)] \geqslant \beta_1 qL_1/4 \tag{4.17}$$

for all $s \in [R^2/2, R^2]$ and $j \in J$.

Proof. By Lemma 4.4, if $\mathfrak{D}^m_R(0) \geqslant L$,

$$P[\mathfrak{D}^m_{r,j}(s;\eta) \geqslant \beta_1 qL] \geqslant 1 - e^{-\beta_1 qL}$$

for all $s \in [R^2/2, R^2]$ and $j \in J$. For $L = c_2/2q$ and large enough c_2, this is at least 1/2. So the analog of (4.17) holds with $\beta_1 qL_1/4$ replaced by $\beta_1 qL/2$ if $\mathfrak{D}^m_R(0) \geqslant L$. Discounting the contribution of $\mathfrak{D}^m_R(0) < L$, it is easy to check that under $E[\mathfrak{D}^m_R(0)] \geqslant L_1$,

$$E[\mathfrak{D}^m_{r,j}(s;\eta)] \geqslant \beta_1 qL_1/2 - \beta_1 c_2/4 \geqslant \beta_1 qL_1/4. \quad \blacksquare$$

By comparing ξ with η, one can now show that one of the following two alternatives must hold: either all cubes $D_{r,j}$ contain (on the average) substantial numbers of both A and B particles at appropriate times s, or the total number of particles in D_R must decrease substantially by time R^2.

Lemma 4.5. Suppose that ξ_0 is translation invariant with $E[\mathfrak{D}^m_R(0)] \geqslant L_1$, where $L_1 \geqslant c_2/q$ for appropriate $c_2 > 0$. Then either

$$E[\mathfrak{D}^m_{r,j}(s;\xi)] \geqslant \beta_1 qL_1/8 \tag{4.18}$$

for all $s \in [R^2/2, R^2]$ and $j \in J$, or

$$E[\mathfrak{D}^T_R(0)] - E[\mathfrak{D}^T_R(R^2;\xi)] \geqslant \beta_1 L_1/8. \tag{4.19}$$

Proof. Suppose the first alternative fails at some $s \in [R^2/2, R^2]$. Then by the above corollary,

$$E[\mathfrak{D}^m_{r,j}(s;\eta)] - E[\mathfrak{D}^m_{r,j}(s;\xi)] \geqslant \beta_1 qL_1/8. \tag{4.20}$$

By the translation invariance of ξ_0 and hence of ξ_s, if (4.20) holds for one j, then it holds for all j. Since $\xi_s \subset \eta_s$,

$$E[\mathfrak{D}^M_{r,j}(s;\eta)] \geqslant E[\mathfrak{D}^M_{r,j}(s;\xi)] \tag{4.21}$$

as well. So by (4.20) and (4.21),

$$E[\mathfrak{D}^T_{r,j}(s;\eta)] - E[\mathfrak{D}^T_{r,j}(s;\xi)] \geqslant \beta_1 qL_1/8. \tag{4.22}$$

Summing over j gives

$$E[\mathfrak{D}^T_R(s;\eta)] - E[\mathfrak{D}^T_R(s;\xi)] \geqslant \beta_1 L_1/8. \tag{4.23}$$

On the other hand,

$$E[\mathfrak{D}^T_R(s;\eta)] = E[\mathfrak{D}^T_R(0)]$$

by the translation invariance of η_0, and

$$E[\mathfrak{D}_R^T(R^2;\xi)] \leqslant E[\mathfrak{D}_R^T(s;\xi)]$$

for $s \leqslant R^2$ by the translation invariance of ξ_0 and the decreasing density of ξ_s. Plugging these into (4.23) gives

$$E[\mathfrak{D}_R^T(0)] - E[\mathfrak{D}_R^T(R^2;\xi)] \geqslant \beta_1 L_1/8. \quad \blacksquare$$

We now set

$$R_t = \sqrt{\delta_1 t} \tag{4.24}$$

and

$$\begin{aligned} r_t &= \delta_2 t^{1/4}, \qquad d<4, \\ &= (\delta_1 t)^{1/d}, \qquad d \geqslant 4, \end{aligned} \tag{4.25}$$

where $\delta_1, \delta_2 > 0$ and will be chosen later. We can assume that δ_1, δ_2 are chosen so r_t divides R_t. Combining Lemmas 4.2 and 4.5, we show in Lemma 4.6 that if $E[\mathfrak{D}_{R_t}^m(0)]$ is not too small, then the total number of particles lost in D_{R_t} over the time interval $[0, R_t^2]$ must also be of this order of magnitude. R_t^2 and r_t^2 stand for $(R_t)^2$ and $(r_t)^2$.

Lemma 4.6. Suppose that ξ_0 is translation invariant with $E[\mathfrak{D}_{R_t}^m(0)] \geqslant L_1$, where $L_1 \geqslant c_2/q$ for appropriate $c_2 > 0$. Then for appropriate $\beta_3 > 0$ (not depending on δ_1, δ_2) and large enough t,

$$E[\mathfrak{D}_{R_t}^T(0)] - E[\mathfrak{D}_{R_t}^T(R_t^2)] \geqslant \beta_3 L_1. \tag{4.26}$$

Proof. Since ξ_0 is translation invariant, ξ_s is translation invariant for all s. So by Lemma 4.2,

$$E[\mathfrak{D}_{r_t,j}^T(s)] - E[\mathfrak{D}_{r_t,j}^T(s+r_t^2)] \geqslant h_d(r_t)\, E[\mathfrak{D}_{r_t,j}^m(s)] \tag{4.27}$$

for all s and $j \in J$. It follows from (4.27) and Lemma 4.5 that either (4.26) holds or

$$E[\mathfrak{D}_{r_t,j}^m(s)] \geqslant \beta_1 q L_1/8 \tag{4.28}$$

for all $s \in [R_t^2/2, R_t^2]$ and $j \in J$. Assume (4.28). Then the right side of (4.27) is at least

$$\beta_1 q L_1 h_d(r_t)/8. \tag{4.29}$$

Letting $s_k \in [R_t^2/2, R_t^2 - r_t^2]$ run through multiples of r_t^2, it follows from (4.27) and (4.29) that

$$E[\mathfrak{D}_{r_t,j}^T(0)] - E[\mathfrak{D}_{r_t,j}^T(R_t^2)] \geqslant \beta_1 q L_1 R_t^2 h_d(r_t)/16 r_t^2.$$

Summing over j, this gives

$$E[\mathfrak{D}_{R_t}^T(0)] - E[\mathfrak{D}_{R_t}^T(R_t^2)] \geqslant \beta_1 L_1 R_t^2 h_d(r_t)/16 r_t^2. \tag{4.30}$$

On the other hand, by Lemma 4.1 and (4.6),

$$\begin{aligned} R_t^2 h_d(r_t)/r_t^2 &\geqslant c_1 R_t^2/r_t^2, && d=1, \\ &\geqslant c_1 R_t^2/r_t^2 \log r_t, && d=2, \\ &\geqslant c_1 R_t^2/r_t^d, && d \geqslant 3. \end{aligned} \tag{4.31}$$

Plugging in R_t and r_t as specified in (4.24) and (4.25), it is easy to check that the right side of (4.31) is c_1 for $d \geqslant 4$ and at least c_1 for $d < 4$ and large enough t. ((4.24) and (4.25) are also used in the proof of Proposition 3.) (4.26) therefore follows from (4.30)–(4.31) with $\beta_3 = \beta_1 c_1/16$. ∎

One can reason as in Lemma 4.6, but instead, after (4.31) plug in other choices of R_t and r_t. Using the bounds thus obtained, one can compute better bounds on $E[\mathfrak{D}_{R_t}^m(s)]$ than needed here; these will be useful in analyzing the spatial configurations of ξ_t for large t (Bramson and Lebowitz[15]).

We are now in a position to demonstrate the main result of the section, Proposition 3, which gives upper bounds for $\rho_A(t)$ in all dimensions. The proposition relies on Lemma 4.6 and two results from Section 2, Corollary 2 of Lemma 2.1 and Lemma 2.2.

Proposition 3. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$ according to (1.1) with $r_A = r_B > 0$. Then

$$\begin{aligned} \rho_A(t) = \rho_B(t) &\leqslant C\sqrt{r_A}/t^{d/4}, && d<4, \\ &\leqslant C(\sqrt{r_A} \vee 1)/t, && d=4, \\ &\leqslant C/t, && d>4, \end{aligned} \tag{4.32}$$

for appropriate C and large enough t.

Proof. Subdivide the interval $[t, 2t]$ into nonoverlapping subintervals of length $\delta_1 t$. (Choose δ_1 so $1/\delta_1$ is integral.) Set $s_k = (1 + \delta_1 k)t$ for $k = 1,\ldots, 1/\delta_1$, and set $I_k = [s_{k-1}, s_k]$. For $R_t = \sqrt{\delta_1 t}$, $s_k = s_{k-1} + R_t^2$. Note also that $\xi_{s_{k-1}}$ is translation invariant. Assume for the moment that

$$E[\mathfrak{D}_{R_t}^m(s_{k-1})] \geqslant L_1 \tag{4.33}$$

for each k and some $L_1 \geq c_2/q$ (with r_t chosen as in (4.25)). It then follows from Lemma 4.6 that for large t,

$$E[\mathfrak{D}^T_{R_t}(s_{k-1})] - E[\mathfrak{D}^T_{R_t}(s_k)] \geq \beta_3 L_1, \tag{4.34}$$

for $k = 1,..., 1/\delta_1$; β_3 does not depend on δ_1. Summing over k, (4.34) gives

$$E[\mathfrak{D}^T_{R_t}(t)] - E[\mathfrak{D}^T_{R_t}(2t)] \geq \beta_3 L_1/\delta_1. \tag{4.35}$$

On the other hand, by Corollary 1 of Lemma 2.1 and Lemma 2.2(a),

$$E[|\mathfrak{D}_{R_t}(s)|] \leq C_1 \sqrt{r_A}\,(R_t)^{d/2} \tag{4.36}$$

for all s and appropriate C_1. Since

$$E[\mathfrak{D}^T_{R_t}(s)] = 2E[\mathfrak{D}^m_{R_t}(s)] + E[|\mathfrak{D}_{R_t}(s)|], \tag{4.37}$$

(4.36) means that

$$E[\mathfrak{D}^T_{R_t}(s)] > 3C_1 \sqrt{r_A}\,(R_t)^{d/2} \Rightarrow E[\mathfrak{D}^T_{R_t}(s)] < 3E[\mathfrak{D}^m_{R_t}(s)]. \tag{4.38}$$

We will show that for appropriate choice of L_1, δ_1, δ_2, (4.35) will be impossible—that is, the drop in $E[\mathfrak{D}^T_{R_t}(s)]$ cannot be as rapid as indicated by (4.35). So the assumption (4.33) must be wrong for at least one s_{k-1}. This will say that $E[\mathfrak{D}^m_{R_t}(s_{k-1})]$ is small relative to $E[\mathfrak{D}^T_{R_t}(t)]$. On account of (4.38), $E[\mathfrak{D}^T_{R_t}(s_{k-1})]$, and hence $E[\mathfrak{D}^T_{R_t}(2t)]$, is also small relative to $E[\mathfrak{D}^m_{R_t}(s_{k-1})]$. So the value of large $E[\mathfrak{D}^T_{R_t}(t)]$ drops quickly between times t and $2t$. One can investigate what "large" means in this context. Division by the volume of D_{R_t} will then give (4.32).

Choose $\delta_1 < \beta_3/12$. We first note that if (4.33) holds with

$$L_1 \equiv E[\mathfrak{D}^T_{R_t}(t)]/12 \geq c_2/q, \tag{4.39}$$

then by (4.35),

$$E[\mathfrak{D}^T_{R_t}(t)] - E[\mathfrak{D}^T_{R_t}(2t)] \geq \beta_3 L_1/\delta_1 > E[\mathfrak{D}^T_{R_t}(t)].$$

This is clearly not possible. So (if $L_1 \geq c_2/q$), (4.33) must be violated with

$$E[\mathfrak{D}^m_{R_t}(s_{k-1})] < E[\mathfrak{D}^T_{R_t}(t)]/12 \tag{4.40}$$

for some $s_{k-1} \in [t, 2t]$. Since $E[\mathfrak{D}^T_{R_t}(s)]$ is decreasing in s, one can apply (4.38) to conclude that

$$E[\mathfrak{D}^T_{R_t}(2t)] \leq E[\mathfrak{D}^T_{R_t}(s_{k-1})] < 3E[\mathfrak{D}^m_{R_t}(s_{k-1})] < E[\mathfrak{D}^T_{R_t}(t)]/4, \tag{4.41}$$

as long as the left side of (4.38) and $L_1 \geqslant c_2/q$ hold. We will iterate (4.41) after investigating (4.38) and $L_1 \geqslant c_2/q$. Note that one can modify δ_1 and L_1 so that the quotient 4 in (4.41) can be replaced by any desired value.

Plugging in the definitions of L_1 and q gives

$$E[\mathfrak{D}^T_{R_t}(t)] \geqslant \frac{12c_2}{q} \geqslant 12c_2(\delta_1)^{d/2}\, t^{d/4}/(\delta_2)^d, \qquad d<4,$$
$$\geqslant 12c_2(\delta_1)^{d/2-1}\, t^{d/2-1}, \qquad d\geqslant 4. \tag{4.42}$$

On the other hand, the left side of (4.38) can be rewritten as (with $s=t$)

$$E[\mathfrak{D}^T_{R_t}(t)] > 3C_1 \sqrt{r_A}\, (\delta_1 t)^{d/4}. \tag{4.43}$$

Choose δ_2 so that

$$\delta_2 \geqslant (\delta_1)^{1/4}(4c_2/C_1\sqrt{r_A})^{1/d}$$

and r_t divides R_t. The right side of (4.43) is then greater than the right side of (4.42) in $d<4$. (4.42) will dominate (4.43) in $d>4$ for large t; in $d=4$, they are of comparable size. So (4.41) will hold for large t as long as

$$E[\mathfrak{D}^T_{R_t}(t)] \geqslant 3C_1\sqrt{r_A}\,(\delta_1 t)^{d/4}, \qquad d<4,$$
$$\geqslant (3C_1\sqrt{r_A} \vee 12c_2)\delta_1 t, \qquad d=4, \tag{4.44}$$
$$\geqslant 12c_2(\delta_1 t)^{d/2-1}, \qquad d>4.$$

(4.44) can be rephrased as

$$\rho_A(t) = \frac{E[\mathfrak{D}^T_{R_t}(t)]}{2(\delta_1 t)^{d/2}} \geqslant C\sqrt{r_A}/2t^{d/4}, \qquad d<4,$$
$$\geqslant C(\sqrt{r_A} \vee 1)/2t, \qquad d=4, \tag{4.45}$$
$$\geqslant C/2t, \qquad d>4,$$

where $C=3C_1(\delta_1)^{-d/4}$ in $d<4$, $C=12c_2(\delta_1)^{-1}$ in $d>4$, and C is chosen correspondingly in $d=4$. Also, (4.41) can be rephrased as

$$\rho_A(2t) < \rho_A(t)/4. \tag{4.46}$$

Denote by $f_d(t)$ the right side of (4.45). For all t,

$$f_d(2t)/f_d(t) \geqslant 1/2.$$

Along the sequence $t=2^n$, $n\geqslant n_0$, large enough n_0, it therefore follows from (4.46) that

$$\rho_A(2t)/f_d(2t) < \tfrac{1}{2}\rho_A(t)/f_d(t) \tag{4.47}$$

as long as $\rho_A(t)/f_d(t) \geq 1$. This ratio therefore falls as n increases until $\rho_A(2^n)/f_d(2^n) < 1$. Since $\rho_A(t)$ is decreasing, $\rho_A(t)/f_d(t) < 2$ for $t \in (2^n, 2^{n+1}]$. Extending this reasoning, it is clear that $\rho_A(t)/f_d(t) < 2$ for large enough t, which demonstrates (4.32). ▮

5. Lower Bounds for Unequal Densities

In this section and the next, we investigate the rate at which $\rho_A(t)$ decays for $r_A < r_B$. The techniques used here are different than for the case $r_A = r_B$, which was covered in the previous sections. On the one hand, for $r_A < r_B$, it is easy to show as in Lemma 5.1 and its corollaries that $\rho_B(t) \to b \equiv r_B - r_A > 0$ as $t \to \infty$. Surviving type-A particles are therefore eventually in an environment consisting almost solely of type-B particles with density approximately b. Since these B particles should be distributed more or less independently, the problem should reduce to the simpler problem of the probability of a particle executing a random walk while avoiding all other random walks up to a given time. As suggested by (1.19), one might expect $\rho_A(t)$ to decay exponentially (even though the exponent given by (1.20) is in fact wrong in low dimensions). On the other hand, it is this rapid decay of $\rho_A(t)$ that causes additional difficulties. One has to worry about a set of small probability when considering the survival of an A particle up to time $s < t$. Conditioned on this unlikely event, the configuration of nearby B particles (or other A particles) could be sharply different than one might expect, enough so to disrupt the above reasoning up to time t. To show that this does not take place, one has to obtain precise estimates.

Here, we demonstrate Proposition 4, which gives lower bounds for $\rho_A(t)$. We will need several lemmas. Lemma 5.1 gives simple bounds on the probability that, in a large cube D_R, the number of A particles or B particles in the noninteracting process η differs by more than a given fraction from its expected value; this probability is exponentially small. The lemma will be used in Section 6 as well to derive estimates on ξ. Lemmas 5.2 and 5.3 justify some of the intuition used above, by showing in an appropriate sense that for large s, the configuration of B particles is dominated by a Poisson random measure with density just slightly greater than b. This is done in two steps: first, that over a suitably large cube the number of B particles at time s will be less than that of the Poisson measure, and then, that this implies that processes starting from these configurations can be coupled so that after an additional amount of time, the configuration of B particles is dominated by the corresponding Poisson measure. Lemma 5.4 gives an upper bound on the number of B particles met by a typical A particle up to a given time for the process η, given appropriate initial densities.

Using these lemmas and a randomization trick, the proof of Proposition 4 is not difficult. We wish to show that $\rho_A(t)$ decays at most exponentially with parameter $\sqrt{t}$ in $d=1$, $t/\log t$ in $d=2$, and t in $d \geqslant 3$. We also wish to show that this rate is proportional to b, except in $d=1$, where the rate involves a different term. (Without this sharpening of the result, the proof is almost trivial in $d \geqslant 3$: one can condition on an A particle not moving and then compute the probability that no B particle ever hits that site. An extension of this reasoning which allows an A particle to move around in a cube of appropriate size also works in $d=1, 2$, cf. ref. 7.) The argument in Proposition 4 consists of taking a typical A particle for the process η and using Lemma 5.4 to compute an upper bound on the number of B particles starting from ξ_0 and on the number of B particles starting from $\xi_{s/2}$, for some appropriate s, that the A particle hits by time t. The first set of B particles is used over times $[0, s]$, and the second over $(s, t]$. Since the density of B particles does not decrease over each of these intervals in the comparison, Lemma 5.4 of course only furnishes us with an upper bound. We can plug the densities provided by Lemma 5.3 into Lemma 5.4 to obtain concrete estimates. The final step involves the realization that we could have doubled the densities of the Poisson random measures dominating the positions of B particles at times 0 and $s/2$. At each point where a B particle hits our selected A particle, one could then discard this B particle with probability 1/2 as being "bogus," that is, coming from the augmented but not the original measures. (This is the reason we need the comparison measure to be Poisson in Lemma 5.3.) The probability that an A particle (in ξ) is not actually hit by a real B particle (and therefore does not disappear) is therefore decreasing not faster than geometrically in the total number (real and bogus) of B particles hit by time t. This gives us Proposition 4 for $d \geqslant 2$. In $d=1$, the event that the number of A particles initially exceeds the number of B particles in spots will be the major contribution to the probability of survival of A particles if r_A is close enough to r_B. This probability is computed in Lemma 5.5. Together with the above estimate, this gives Proposition 4 in $d=1$ as well.

Lemma 5.1 and its corollaries are elementary observations. Below, $m_A(R)$ and $m_B(R)$ denote the mean number of A and B particles in D_R at time $0-$.

Lemma 5.1. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$ according to a homogeneous Poisson measure as in (1.1). Then

$$P\left[\mathfrak{D}_R(t;\xi) \leqslant \frac{1}{2}(m_B(R) - m_A(R))\right],$$

$$P\left[\mathfrak{D}_R(t;\xi) \geqslant \frac{3}{2}(m_B(R) - m_A(R))\right] \leqslant \exp\left\{-\frac{(m_B(R) - m_A(R))^2}{24 m_B(R)}\right\}.$$

Proof. Using Lemma 2.1 and the convexity of $\exp\{\theta x\}$, we compare the processes ξ and η as follows:

$$E[\exp\{\theta[\mathfrak{D}_R(t;\xi)-(m_B(R)-m_A(R))]\}]$$
$$\leqslant E[\exp\{\theta[\mathfrak{D}_R(t;\eta)-(m_B(R)-m_A(R))]\}].$$

Since particles do not interact for the η process, the initial measures are invariant for η. Since the initial measures are also independent, a simple computation shows the right side equals

$$\exp\{m_B(R)(e^{\theta}-\theta-1)\}\exp\{m_A(R)(e^{-\theta}+\theta-1)\}.$$

So Chebyshev's inequality shows that for $\theta>0$,

$$P[\mathfrak{D}_R(t;\xi)-(m_B(R)-m_A(R))\geqslant \tfrac{1}{2}(m_B(R)-m_A(R))]$$
$$\leqslant \exp\{m_B(R)e^{\theta}+m_A(R)e^{-\theta}$$
$$-\tfrac{3}{2}(m_B(R)-m_A(R))\theta-(m_B(R)+m_A(R))\}.$$

For

$$\theta=\frac{1}{2}\frac{m_B(R)-m_A(R)}{m_B(R)+m_A(R)},$$

a little estimation by expanding up to the second term in the Taylor series shows this is

$$\leqslant \exp\left\{-\frac{1}{12}\frac{(m_B(R)-m_A(R))^2}{m_B(R)+m_A(R)}\right\}\leqslant \exp\left\{-\frac{(m_B(R)-m_A(R))^2}{24m_B(R)}\right\}. \quad (5.1)$$

By reasoning as above but with $\exp\{-\theta x\}$ instead, one obtains the same expression for

$$P[\mathfrak{D}_R(t;\xi)-(m_B(R)-m_A(R))\leqslant -\tfrac{1}{2}(m_B(R)-m_A(R))]$$

after expanding e^{θ}. The bound (5.1) therefore holds again. ∎

Corollary 1. Assume initial conditions as in Lemma 5.1 with $r_A<r_B$. Then

$$\rho_A(t)\to 0, \quad \rho_B(t)\to\rho_B(0)-\rho_A(0) \quad \text{as} \quad t\to\infty. \quad (5.2)$$

Proof. Since $\rho_B(t)-\rho_A(t)$ is constant in t, the first limit implies the second. Suppose now that $\rho_A(t)\geqslant\delta>0$ for all t. On account of the first inequality of Lemma 5.1, if R is chosen large enough, then with probability at least $\delta/2$ the cube D_R contains both A and B particles at time t. Since with some probability q_R at least one pair of these particles will hit by time $t+1$,

$$\rho_A(t) - \rho_A(t+1) \geqslant \delta q_R/2R^d. \tag{5.3}$$

Repeatedly applying (5.3) gives a contradiction. ∎

Note that one can also show (5.2) in the same basic way by using the ergodic theorem instead of Lemma 5.1. Also, rates at which $\rho_A(t) \to 0$ are obtained in Section 6 by examining the structure of particle configurations in appropriate D_R. Using the above lemma and corollary, we see that:

Corollary 2. Assume initial conditions as in Lemma 5.1 with $r_A < r_B$. As $t,\ R \to \infty$,

$$P[\mathfrak{D}_R^B(t;\xi) \geqslant 2(m_B(R) - m_A(R))] \to 0. \tag{5.4}$$

Proof. Since $\mathfrak{D}_R^B = \mathfrak{D}_R + \mathfrak{D}_R^A$,

$$\begin{aligned} &P[\mathfrak{D}_R^B(t;\xi) \geqslant 2(m_B(R) - m_A(R))] \\ &\quad \leqslant P[\mathfrak{D}_R(t;\xi) \geqslant \tfrac{3}{2}(m_B(R) - m_A(R))] \\ &\qquad + P[\mathfrak{D}_R^A(t;\xi) \geqslant \tfrac{1}{2}(m_B(R) - m_A(R))]. \end{aligned} \tag{5.5}$$

The first term on the right $\to 0$ independently of t as $R \to \infty$ by Lemma 5.1. The second term $\to 0$ independently of R as $t \to \infty$ by Corollary 1 and Markov's inequality. ∎

In Lemma 5.3, we will need to show that an appropriate Poisson measure dominates a measure, the location of whose particles is not completely prescribed. For this, we will use Lemma 5.2, which says that if a particle in $\mathbb{Z}^d$ is surrounded on all $2d$ sides by other particles not too far away, then the random walk executed by this particle may be coupled with the random walks executed by the other particles so that after a large enough (deterministic) time t_0, this particle is always at a site occupied by at least one of the other particles. (The result is given by Theorem 6 in Bramson and Griffeath[16]). To be more precise, introduce the following notation. Let

$$I = \{i = (i_1, \ldots, i_d) \text{ with } i_k = \pm 1\}. \tag{5.6}$$

Denote by D_R^i, $i \in I$, the cube

$$D_R^i = D_R + Ri. \tag{5.7}$$

As usual, we let $Y_t^{y^{i,l}}$ denote independent random walks with $Y_0^{y^{i,l}} = y^{i,l}$. $Y_t^{y^0}$ will be another random walk which we wish to couple to the other random walks.

Lemma 5.2. Assume that $y^0 \in D_R$ and $y^{i,l} \in D_R^i$ for all $i \in I$, $1 \leqslant l \leqslant 4$. Then Y^{y^0} can be coupled to $(Y^{y^{i,l}};\ i \in I,\ 1 \leqslant l \leqslant 4)$ so that for $t \geqslant C_1 R^2$, some appropriate $C_1 > 0$,

$$Y_t^{y^0} = Y_t^{y^{i,l}} \qquad \text{for some} \quad i \in I,\ l \in \{1, 2, 3, 4\} \tag{5.8}$$

on a set of probability 1.

For Lemma 5.3, we use $D_{R,j}$, $j \in J_M$, to denote the disjoint translates of D_R which cover D_M, where R divides M. We choose $R = Q\sqrt{s}$ and $M = 2N\sqrt{s}$ for appropriate Q and N. As earlier, the norm $\|x\|$ is chosen so $x \in D_R$ iff $\|x\| \leqslant R$. Let λ and μ be random configurations whose states consist of finite numbers of particles at each site in $\mathbb{Z}^d$. We say that λ dominates μ (on a set F) if it is possible to couple λ and μ so that $\mu \subset \lambda$ for all ω $(\in F)$.

We also need to introduce two new processes. ξ_t^B will denote the restriction of our two-particle annihilating random walk at time t to the B particles present (i.e., ignore the A particles). $\eta_t^{\mathscr{A},s}$, $t \geqslant s$, will denote the process consisting of independent random walks with $\eta_s^{\mathscr{A},s} = \mathscr{A}$, where $\mathscr{A}$ is a random configuration of particles in $\mathbb{Z}^d$. We will in particular be interested in the case where $\mathscr{A} = \mathscr{P}$, a Poisson measure with density $\mathscr{P}(x)$ at $x \in \mathbb{Z}^d$. ($\mathscr{P}$ as used here is not to be confused with the percolation substructure $\mathscr{P}$.) The processes $\eta_t^{\mathscr{P},s}$ will typically be coupled to ξ_t^B for $t \geqslant s$. In keeping with our previous notation, $\mathfrak{D}_{R,j}(t;\cdot)$ will denote the (signed) number of particles in $D_{R,j}$ for the corresponding process.

We now demonstrate Lemma 5.3. It says that at large enough t, one can dominate ξ_t^B by an appropriate Poisson measure with low density.

Lemma 5.3. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$ according to (1.1) with $r_A < r_B$. Then for all N, there exists t_N (also depending on r_A, r_B) so that for $t \geqslant t_N$, ξ_t^B is dominated on a set of probability at least 7/8 by a Poisson measure $\mathscr{P}$ with

$$\begin{aligned} \mathscr{P}(x) &= 2^{d+5}(r_B - r_A) && \text{for} \quad \|x\| \leqslant N\sqrt{t}, \\ &= 2r_B && \text{for} \quad \|x\| > N\sqrt{t}. \end{aligned} \tag{5.9}$$

Note that it should of course be possible to show the analog of (5.9) with the coefficient of $(r_B - r_A)$ inside the cube as close to 1 as desired and outside the cube $= 1$. The authors could almost show this but got tired of trying. The point is that if $r_B - r_A$ is small, then so is $\mathscr{P}(x)$ for $\|x\| \leqslant N\sqrt{t}$.

Proof of Lemma 5.3. We first note that we can assume that $2^{d+5}(r_B - r_A) \leqslant 2r_B$, since otherwise the bound is trivial. By Corollary 2 of Lemma 5.1, for each $\varepsilon > 0$ and large enough R and s,

$$P[\mathfrak{D}_R(s; \xi^B) \geqslant 2(m_B(R) - m_A(R))] < \varepsilon. \tag{5.10}$$

Set $\varepsilon = \varepsilon_1/(2N/Q)^d$, where $\varepsilon_1 = 1/16$. Then

$$P[\mathfrak{D}_{R,j}(s; \xi^B) \geqslant 2(m_B(R) - m_A(R)) \text{ for some } j \in J_M] < 1/16 \tag{5.11}$$

for $s \geqslant t_N$.

We will now compare ξ_t^B, $t \geqslant s$, with the process $\eta_t^{\mathscr{P}_1, s}$ with Poisson measure $\mathscr{P}_1$,

$$\begin{aligned} \mathscr{P}_1(x) &= 2^{d+4}(r_B - r_A) \qquad \text{for} \quad \|x\| \leqslant M, \\ &= 2r_B \qquad \text{for} \quad \|x\| > M. \end{aligned} \tag{5.12}$$

It is a simple large deviations estimate (similar to that used in Lemma 5.1) that for large enough R,

$$P[\mathfrak{D}_{R,j}(s; \eta^{\mathscr{P}_1, s}) \leqslant 2^{d+3}(m_B(R) - m_A(R)) \text{ for some } j \in J_M] < 1/16. \tag{5.13}$$

By (5.11) and (5.13),

$$P[\mathfrak{D}_{R,j_1}(s; \eta^{\mathscr{P}_1, s}) \geqslant 2^{d+2}\mathfrak{D}_{R,j_2}(s; \xi^B), \forall j_1, j_2 \in J_M] \geqslant 7/8. \tag{5.14}$$

Denote the set on which this inequality holds by G. For each B particle at $y^0 \in D_{R,j} \cap \xi_s^B$, $j \in J_M$, one can therefore choose 2^{d+2} random walks at $y^{i,l} \in \eta_s^{\mathscr{P}_1, s}$, $i \in I$, $1 \leqslant l \leqslant 4$, where, using the notation above Lemma 5.2,

$$y^{i,l} \in D'_{R,j} = D_{R,j} + Ri. \tag{5.15}$$

These 2^{d+2} random walks move independently. By choosing Q $(= R/\sqrt{s})$ not too large, one can therefore employ Lemma 5.2 to couple the B particle at y^0 with the random walks at $y^{i,l}$ so that after further time s, this particle will (if it has not yet disappeared) occupy a site occupied by one of the corresponding random walks from $\eta^{\mathscr{P}_1, s}$. The 2^{d+2} random walks corresponding to each B particle in $D_M \cap \xi_s^B$ are assumed to be distinct; this coupling can therefore be performed simultaneously over all such B particles. On the other hand, $\xi_s^B \subset \eta_s^B$, the number of B particles from the corresponding noninteracting process. We will use this for B particles at $y^0 \in \xi_s^B - D_M$. η_s^B is Poisson distributed with constant mean r_B. One can therefore use it and an independent copy on D_M^c to construct $\mathscr{P}_1$ there. This gives an obvious coupling of ξ_s^B to $\eta_s^{\mathscr{P}_1, s}$ for particles outside

D_M. The independent copy of η_s^B on D_M^c is used to contribute random walks used in the first part of the construction for coupling in the cases where $y^0 \in D_{R,j} \cap \xi_s^B$, but $D_{R,j}^i \subset D_M^c$.

We can therefore couple the processes ξ_t^B and $\eta_t^{\mathscr{P}_1,s}$ starting at time s on G so that: (1) each B particle at $y^0 \in D_M \cap \xi_s^B$ occupies the same site as one of the corresponding 2^{d+2} random walks at $y^{i,l} \in \eta_s^{\mathscr{P}_1,s}$ for all $t \geqslant 2s$ and (2) each B particle at $y^0 \in \xi_s^B - D_M$ occupies the same site as the corresponding random walk starting in $\mathscr{P}_1$ for all $t \geqslant s$. B particles from ξ_t^B may disappear, but not random walks from $\eta_t^{\mathscr{P}_1,s}$. So for each particle present at ξ_s^B on G at time s and still existing by time $t \geqslant 2s$, there is a corresponding particle from $\eta_t^{\mathscr{P}_1,s}$ at the same site. We have therefore shown that for $s \geqslant t_N$ and $t \geqslant 2s$,

$$P[\xi_t^B \subset \eta_t^{\mathscr{P}_1,s}] \geqslant 7/8. \tag{5.16}$$

We will apply (5.16) at $t = 2s$.

To finish, note that since $\eta_s^{\mathscr{P}_1,s}$ is Poisson, so is $\eta_{2s}^{\mathscr{P}_1,s}$. Also,

$$\begin{aligned} E[\eta_{2s}^{\mathscr{P}_1,s}(x)] &= \sum_{y \in D_M} E[\eta_s^{\mathscr{P}_1,s}(y)]\, P[Y_s^y = x] + \sum_{y \notin D_M} E[\eta_s^{\mathscr{P}_1,s}(y)]\, P[Y_s^y = x] \\ &\leqslant 2^{d+4}(r_B - r_A) + 2r_B \sum_{y \notin D_M} P[Y_s^y = x]. \end{aligned} \tag{5.17}$$

For $x \in D_{N\sqrt{2s}}$ and $y \notin D_M$, $\|x - y\| \geqslant N\sqrt{s}/2$. By choosing N large enough, the last sum in (5.17) can be made as small as desired. So for large enough N and $s \geqslant t_N$, and $x \in D_{N\sqrt{2s}}$,

$$E[\eta_{2s}^{\mathscr{P}_1,s}(x)] \leqslant 2^{d+5}(r_B - r_A). \tag{5.18}$$

Since the larger N, the stronger the statement, (5.18) holds for all N. On the other hand,

$$E[\eta_{2s}^{\mathscr{P}_1,s}(x)] \leqslant 2r_B \tag{5.19}$$

is always true. So $\eta_{2s}^{\mathscr{P}_1,s}$ is Poisson with densities bounded as in (5.18) and (5.19). The conclusion of the lemma follows from this and (5.16). ∎

Lemma 5.4 says that up to a given time, a typical random walk Y_s^y will not meet more than a specific number of other random walks, this amount depending on the initial densities given. More specifically, we consider the process η_s of noninteracting random walks with η_0 given by a (not necessarily Poisson) random initial measure with density $r(x)$ at each $x \in \mathbb{Z}^d$. Add to this another independent random walk Y_s^y with $Y_0^y = y$. Let

$\mathfrak{N}(t)=$ the number of random walks (from η) Y_s^y meets by time t. Then the following holds. Recall that

$$\begin{aligned} g_d(t) &= \sqrt{t}, & d&=1, \\ &= t/\log t, & d&=2, \\ &= t, & d&\geqslant 3. \end{aligned} \tag{5.20}$$

Lemma 5.4. Suppose that

$$\begin{aligned} r(x) &\leqslant r && \text{for} \quad \|x\| \leqslant N\sqrt{t}, \\ &\leqslant r' && \text{for} \quad \|x\| > N\sqrt{t}. \end{aligned} \tag{5.21}$$

Then, for $t \geqslant t_0$, $\|y\| \leqslant N/2$ and $N \geqslant N_0$ (depending on r, r'), and appropriate C_2,

$$P[\mathfrak{N}(t) \geqslant C_2 r g_d(t)] < 1/4. \tag{5.22}$$

Proof. One can give a proof by reasoning similar to that in Lemma 4.1 by generalizing the bounds on the function G_t, obtaining estimates similar to (4.4) but for all x and in the opposite direction, and then integrating. Instead, we present the following argument. First, introduce the following notation. Let $l(s)$ be the number of random walks of η at Y_s^y at time s. Let T_k denote the time at which the kth distinct random walk and Y^y first meet; we order the random walks so that $T_1 \leqslant T_2 \leqslant \cdots$. Also, let $L_k(s)$ denote the amount of time spent by Y^y and this kth random walk at a common site up to time s.

Since $\|y\| \leqslant N/2$, it is easy to see that if $s \in [0, 2t]$ and $N \geqslant N_0$, some appropriate N_0, then

$$E[l(s)] \leqslant 2r;$$

consequently,

$$E\left[\int_0^{2t} l(s)\, ds\right] \leqslant 4rt. \tag{5.23}$$

On the other hand,

$$\begin{aligned} E\left[\int_0^{2t} l(s)\, ds\right] &= E\left[\sum_k L_k(2t)\right] \\ &\geqslant \sum_k E[L_k(2t);\, T_k \leqslant t]. \end{aligned} \tag{5.24}$$

Now,

$$E[L_k(2t);\, T_k \leqslant t] \geqslant P[T_k \leqslant t]\; G_t(0), \tag{5.25}$$

where $G_t(0)$ is defined in (4.3) and is the expected occupation time at 0 up to time t spent by a rate-2 random walk starting at 0. As mentioned in Lemma 4.1,

$$G_t(0) \approx \beta'_d t/g_d(t) \tag{5.26}$$

for large t. (The reasoning for $d=1$ is the same.) So putting (5.23)–(5.26) together produces

$$4rt \geqslant (\beta'_d t/g_d(t)) \sum_k P[T_k \leqslant t] = (\beta'_d t/g_d(t))\, E[\mathfrak{N}(t)].$$

So

$$E[\mathfrak{N}(t)] \leqslant (4/\beta'_d)\, r g_d(t). \tag{5.27}$$

(5.22) follows from (5.27) and Markov's inequality. ∎

In Lemma 5.5, we show that if r_A is close enough to r_B, then in $d=1$ the event that the number of A particles initially locally exceeds the number of B particles will be large enough to entail a comparatively large survival probability of A particles.

Lemma 5.5. Assume initial data as in (1.1) in $d=1$ with $3r_B/4 < r_A < r_B$. Then for appropriate C_3 and large t (depending on r_A, r_B),

$$\rho_A(t) \geqslant \exp\{-C_3\sqrt{t}\,(r_B - r_A)^2/r_B\}. \tag{5.28}$$

Proof. $\mathfrak{D}^A_M(0-)$, $\mathrm{D}^B_M(0-)$ are Poisson distributed with means $m_A(M)$ and $m_B(M)$. (M will be chosen to grow proportional to $\sqrt{t}$.) We can apply Stirling's formula to show, with a little computation, that

$$\begin{aligned} &P[\mathfrak{D}^B_M(0-) = [2m_A(M) - m_B(M)]] \\ &\quad \geqslant C_4 \exp\{2(m_A - m_B)\} \left(\frac{m_B}{2m_A - m_B}\right)^{2m_A - m_B} \Big/ \sqrt{2\pi(2m_A - m_B)} \end{aligned} \tag{5.29}$$

for appropriate $C_4 > 0$, where we abbreviate $m_A(M)$ ($m_B(M)$) by m_A (m_B) and $[x]$ means the integral part of x. (For simplicity, we assume M is large enough so $m_B - 1 \approx m_B$.) The right side of (5.29)

$$\begin{aligned} = C_4 \exp\Big\{&2(m_A - m_B) + (2m_A - m_B) \\ &\times \log\left(1 + \frac{2(m_B - m_A)}{2m_A - m_B}\right)\Big\} \Big/ \sqrt{2\pi(2m_A - m_B)}. \end{aligned}$$

Using the inequality $\log(1+x) \geqslant x - x^2/2$, $x \geqslant 0$, and $3r_B/4 < r_A < r_B$, we see that this is at least

$$C_4 \exp\{-4(m_B(M) - m_A(M))^2/m_B(M)\}/\sqrt{2\pi m_B(M)}.$$

Of course, this bound is what one would expect by plugging in a normal approximation for the Poisson distribution. Also, note that

$$P[\mathfrak{D}_M^A(0-) = [m_A(M)] + 1] \approx \frac{1}{\sqrt{2\pi m_A(M)}} \tag{5.30}$$

for large M. Let

$$F = \{\mathfrak{D}_M^A(0-) = [m_A(M)] + 1, \mathfrak{D}_M^B(0-) = [2m_A(M) - m_B(M)]\}. \tag{5.31}$$

$P(F)$ is given by the product of the probabilities in (5.29) and (5.30); for our purposes, this is a large enough event to work with. We consider the process ξ with ξ_{0-} conditioned by assuming that F holds; denote the new process by ξ^F. Also, let η^F be the corresponding process of independent random walks. We will show that for appropriate M, $E[\mathfrak{D}_M(t; \xi^F)]$ is bounded below and away from 0. The density of A particles in D_M will therefore not be too small and this will give us a bound on $\rho_A(t)$.

To see this, set

$$m_s^F(M) = -E[\mathfrak{D}_M(s; \xi^F)] = -E[\mathfrak{D}_M(s; \eta^F)],$$

where $M = N\sqrt{t}$ and N is fixed. Of course,

$$\begin{aligned} m_0^F(M) &= [m_A(M)] - [2m_A(M) - m_B(M)] + 1 \\ &\geqslant m_B(M) - m_A(M) = N\sqrt{t}\,(r_B - r_A). \end{aligned} \tag{5.32}$$

It is easy to choose N large enough so that not enough random walks have crossed the boundary of D_M by time t to change the value of $m_t^F(M)$ much. (Recall that η_0^F outside of D_M has constant density.) We can therefore choose N so that

$$m_t^F(M) \geqslant m_0^F(M)/2 \geqslant (m_B(M) - m_A(M))/2. \tag{5.33}$$

Consequently,

$$E[\mathfrak{D}_M^A(t; \xi^F)] \geqslant (m_B(M) - m_A(M))/2. \tag{5.34}$$

So for the process ξ,

$$\begin{aligned} E[\mathfrak{D}_M^A(t; \xi)] \geqslant{}& \frac{C_5}{m_B(M)} (m_B(M) - m_A(M)) \\ &\times \exp\{-4(m_B(M) - m_A(M))^2/m_B(M)\} \end{aligned}$$

by (5.29)–(5.30) for appropriate $C_5 > 0$. Sinice ξ is translation invariant, this implies

$$\rho_A(t) \geqslant C_5 \frac{(r_B - r_A)}{r_B} \exp\{-4N\sqrt{t}\,(r_B - r_A)^2/r_B\}.$$

(5.28) follows for appropriate choice of C_3. ∎

One can now follow the outline presented at the beginning of the section to prove Proposition 4. Most of the work is to compute the dependence of $\rho_A(t)$ on r_A, r_B.

Proposition 4. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$ according to (1.1) with $0 < r_A < r_B$. Then for $g_d(t)$ defined as in (5.20),

$$\begin{aligned} \rho_A(t) &\geqslant \exp\{-A((r_B - r_A)^2/r_B)\, g_1(t)\}, && d = 1, \\ &\geqslant \exp\{-A(r_B - r_A)\, g_d(t)\}, && d \geqslant 2, \end{aligned} \qquad (5.35)$$

for appropriate A (depending on d) and large enough t.

Proof. Before beginning the main of the proof, we modify the above statement slightly. It suffices to instead show that a random walk Y_s^0, with $Y_0^0 = 0$, does not meet any B particles by time t with a probability at least as great as that given on the right side of (5.35), with some $A' < A$ substituted for A. (We are in effect inserting an extra A particle into the system, which could conceivably alter the dynamics.) To see this, note that A particles can be broken into two groups with Poisson random measures and with densities $\varepsilon > 0$ and $r_A - \varepsilon$. Y_s^0 can be chosen from the first group with all other such A particles being discarded; the A particles from the second group are assumed to evolve and to interact with B particles as usual. After computing the probability that Y_s^0 meets no B particles for this modified system, we can then invoke the "maximum principle," Lemma 3.2, to justify the same bound for Y_s^0 meeting no B particles in the original system.

We restrict our attention to

$$G_1 = \{\|Y_s^0\| \leqslant N_1\sqrt{t} \text{ for } s \in [0, t]\};$$

one can choose N_1 large enough so that

$$P[G_1] \geqslant 7/8. \qquad (5.36)$$

One is interested in seeing to what extent Y_s^0 restricted to G_1 must hit random walks starting from certain configurations at times $0-$ and a given

t_1. To be more specific, let $\mathscr{P}_1, \mathscr{P}_2$ be Poisson measures with densities $\mathscr{P}_1(x) \equiv 2r_B$ and

$$\begin{aligned} \mathscr{P}_2(x) &= 2^{d+6}(r_B - r_A) \qquad \text{for} \quad \|x\| \leqslant 2N_1\sqrt{t-t_1}, \\ &= 4r_B \qquad \text{for} \quad \|x\| > 2N_1\sqrt{t-t_1}, \end{aligned}$$

where $t_1 = \delta t$, $0 < \delta \leqslant 1/2$, will be chosen later. We set $t_2 = t - t_1$. Consider the processes of independent random walks $\eta_s^{\mathscr{P}_1, 0-}$, $s \in [0, t_1]$, and $\eta_s^{\mathscr{P}_2, t_1}$, $s \in [t_1, t]$, introduced before Lemma 5.3. Let $\mathfrak{N}_1(t_1)$ and $\mathfrak{N}_2(t)$ denote the number of random walks from each process that Y_s^0 ultimately meets. By Lemma 5.4 with $r = r' \equiv 2r_B$, if $t_1 \geqslant t_0$, then

$$P[\mathfrak{N}_1(t_1) \geqslant f_1(t_1)] < 1/4, \tag{5.37}$$

where

$$f_1(t_1) = 2C_2 r_B\, g_d(t_1).$$

One can also apply Lemma 5.4 with $N = 2N_1$,

$$r = 2^{d+6}(r_B - r_A), \qquad r' = 4r_B,$$

to conclude that for N_1 large enough,

$$P[\mathfrak{N}_2(t) \geqslant f_2(t_2); G_1] < 1/4, \tag{5.38}$$

where

$$f_2(t_2) = 2^{d+6} C_2(r_B - r_A)\, g_d(t_2).$$

We now consider the following "thinning" procedure for $\mathscr{P}_1$ and $\mathscr{P}_2$. For every random walk in $\mathscr{P}_1$ at time $0-$, flip a fair coin. If it comes up heads, keep the random walk, otherwise discard it immediately. Denote by $\mathscr{P}'_1$ the measure of such remaining random walks at time $0-$. We can consider these remaining random walks as "authentic," and the discarded random walks as "bogus." $\mathscr{P}'_1$ is Poisson with $\mathscr{P}'_1(x) \equiv r_B$; it has the same distribution as ξ_{0-}^B. Particles in $\eta^{\mathscr{P}'_1, 0-}$ just execute random walks, whereas those in ξ^B can disappear (by hitting A particles). One can assume without loss of generality that the processes are coupled so that particles move together and $\xi_{0-}^B = \mathscr{P}'_1$; then,

$$\xi_s^B \subset \eta_s^{\mathscr{P}'_1, 0-} \qquad \text{for all} \quad s \in [0, t_1]. \tag{5.39}$$

One can follow the same procedure for $\mathscr{P}_2$, and denote by $\mathscr{P}'_2$ the measure of remaining random walks at time t_1. $\mathscr{P}'_2$ is Poisson with

$$\begin{aligned} \mathscr{P}'_2(x) &= 2^{d+5}(r_B - r_A) \qquad \text{for} \quad \|x\| \leqslant 2N_1\sqrt{t_2}, \\ &= 2r_B \qquad \text{for} \quad \|x\| > 2N_1\sqrt{t_2}. \end{aligned} \tag{5.40}$$

On account of Lemma 5.3, $\mathscr{P}'_2$ dominates the measure $\xi^B_{t_1}$ on a set of probability 7/8 for $t = t_1/\delta \geqslant t_N/\delta$ with

$$N = 2N_1\sqrt{t_2/t_1} = 2N_1\sqrt{(1-\delta)/\delta}.$$

That is, $\mathscr{P}'_2$ can be coupled to $\xi^B_{t_1}$ so that on an appropriate set G_2 with

$$P[G_2] \geqslant 7/8, \tag{5.41}$$

$\xi^B_{t_1} \subset \mathscr{P}'_2$. As above, particles in ξ^B can disappear whereas particles in $\eta^{\mathscr{P}'_2, t_1}$ do not. One can therefore couple the processes so that on G_2,

$$\xi^B_s \subset \eta^{\mathscr{P}'_2, t_1}_s \qquad \text{for all} \quad s \in [t_1, t]. \tag{5.42}$$

We now put everything together. Let $\mathfrak{N}(t)$ denote the total number of random walks from both $\eta^{\mathscr{P}_1, 0-}$ and $\eta^{\mathscr{P}_2, t_1}$ over the intervals $[0, t_1]$ and $[t_1, t]$ that Y^0_s ultimately meets. Also, let $\mathfrak{N}'(t)$ ($\mathfrak{N}^\xi(t)$) be the number of random walks (B particles) from $\eta^{\mathscr{P}'_1, 0-}$ and $\eta^{\mathscr{P}'_2, t_1}$ ($\xi^B_{t_1}$) that Y^0_s meets. On account of (5.39) and (5.42),

$$\mathfrak{N}^\xi(t) \leqslant \mathfrak{N}'(t) \quad \text{on} \quad G_2. \tag{5.43}$$

We wish to compute a lower bound for $P[\mathfrak{N}^\xi(t) = 0]$ to obtain (5.35). By (5.37) and (5.38),

$$P[\mathfrak{N}(t) > f_1(t_1) + f_2(t_2); G_1] < 1/2. \tag{5.44}$$

Since the choice of authentic/bogus particles is independent of everything else in the processes,

$$\begin{aligned} P[\mathfrak{N}^\xi(t) = 0] &\geqslant P[\mathfrak{N}'(t) = 0; G_2] \\ &\geqslant P[\mathfrak{N}(t) \leqslant f_1(t_1) + f_2(t_2); G_1 \cap G_2]\, 2^{-(f_1(t_1) + f_2(t_2))} \\ &\geqslant (\tfrac{1}{2} - P[(G_1 \cap G_2)^c])\, 2^{-(f_1(t_1) + f_2(t_2))}. \end{aligned} \tag{5.45}$$

On account of (5.36) and (5.41), this is

$$\geqslant 2^{-(f_1(t_1) + f_2(t_2) + 2)}. \tag{5.46}$$

So by (5.45)–(5.46),

$$P[\mathfrak{N}^\xi(t) = 0] \geqslant 2^{-(f_1(t_1) + f_2(t_2) + 2)}. \tag{5.47}$$

On the other hand,

$$f_1(t_1) + f_2(t_2) + 2 = 2C_2 r_B\, g_d(t_1) + 2^{d+6} C_2 (r_B - r_A)\, g_d(t_2) + 2. \tag{5.48}$$

Since $t_1 + t_2 = t$, it is not difficult to check that for small enough δ (depending on r_A, r_B), $g_d(t_1)$ will be sufficiently small so that (5.48) is

$$\leqslant 2^{d+7} C_2(r_B - r_A)\, g_d(t). \tag{5.49}$$

By (5.47)–(5.49) with $A' = 2^{d+7} C_2 \log 2$,

$$P[\mathfrak{N}^{\xi}(t) = 0] \geqslant \exp\{-A'(r_B - r_A)\, g_d(t)\}. \tag{5.50}$$

(5.50) is the desired bound. As noted in the beginning of the proof, this implies (5.35) for $d \geqslant 2$ with $A > A'$. For $d = 1$, note that Lemma 5.5 already gives us (5.35) for $r_A > 3r_B/4$. For $r_A \leqslant 3r_B/4$, (5.50) works if we increase A by a factor of 4. (One can also give a much simpler argument in this case by computing the probability that $\eta_s^B \cap D_{\sqrt{t}} = \varnothing$ for all $s \in [0, t]$.) ∎

6. *Upper Bounds for Unequal Densities, $d > 1$*

In this section and the next, we give upper bounds on $\rho_A(t)$ for $r_A < r_B$; these correspond to the lower bounds given in Section 5. Here, we need to show that A and B particles are sufficiently randomly distributed so that a typical A particle "feels" the greater density of B particles. Then, presumably, the decay of $\rho_A(t)$ should be exponential. As might be expected from the somewhat different nature of the results, the reasoning for $d > 1$ and $d = 1$ differs. We will do the case $d = 1$ in Section 7.

The Basic Idea; Lemmas 6.1 and 6.2

In Lemmas 6.1 and 6.2, we give lower bounds on the number of other random walks (starting from a fixed concentration) a random walk will typically hit by time t. In Lemma 6.2, this number will be of order $g_d(t)$ (already defined in (1.22)),

$$\begin{aligned} g_d(t) &= \sqrt{t}, && d = 1, \\ &= t/\log t, && d = 2, \\ &= t, && d \geqslant 3. \end{aligned} \tag{6.1}$$

The probability that substantially fewer random walks will be hit will be exponentially small with exponent of order $g_d(t)$. If one multiplies this exponent by $r_B - r_A$, the ultimate density of the B particles, and then exponentiates, one obtains the bound in Proposition 5. The dependence on t but not on the initial densities is also correct in $d = 1$. On the other hand, it should presumably be the case that if an A particle hits on the order of $g_d(t)$ B-particles when annihilation is suppressed, then this A particle should in fact meet a B particle, and thus disappear. It therefore makes some sense that the probability that an A particle survives is bounded by this exceptional probability. Unfortunately, in a rigorous argument one needs to pay particular attention to the possibility that A particles could conceivably cluster (they do in $d=1$). Such clustering might allow the A particles to "protect" each other, thereby interfering with the above scenario and slowing down the rate at which A particles meet B particles. We were therefore not able to obtain a direct argument using the lemma. Instead, Lemmas 6.3–6.6 give an indirect argument based on iteration. The rate obtained in this manner is much slower than the actual rate, but still suffices when used together with Lemma 6.2. This is carried out in the proof of Proposition 5. Unless stated otherwise, $d>1$ will be assumed for the remainder of the section.

Lemma 6.1 says basically that if random walks $Y_s^{y_j}$, $j\in\mathcal{J}$, satisfy an approriate initial concentration relative to a given path $y(s)$, then the probability $y(s)$ meets at least one of them by time t is at least a fixed multiple of

$$\begin{aligned} f_d(t) &= 1/\log t, & d&=2,\\ &= t^{1-d/2}, & d&\geqslant 3; \end{aligned} \tag{6.2}$$

let T be this hitting time. $D_{R,j}$, $j\in\mathbb{Z}^+$, will denote disjoint translates of the cube D_R which partition $\mathbb{Z}^d$. $\mathcal{J}$ will be the set of j where $y(s)\in D_{R,j}$ for some $s\in[0,t]$.

Lemma 6.1. Assume that $y_j\in D_{\sqrt{t},j}$ for $j\in\mathcal{J}$. Then for appropriate $c_1>0$ and large enough t (not depending on $y(\cdot)$ or y_j),

$$P[T\leqslant t]\geqslant c_1 f_d(t). \tag{6.3}$$

Proof. Let G_s denote the event that $Y_s^{y_j}=y(s)$ for some $j\in\mathcal{J}$. We will show that

$$P[T\leqslant t]=\frac{E[\int_0^t 1_{G_s}\,ds]}{E[\int_0^t 1_{G_s}\,ds\mid T\leqslant t]}=\frac{\int_0^t P[G_s]\,ds}{\int_0^t P[G_s\mid T\leqslant t]\,ds} \tag{6.4}$$

is at least $c_1 f_d(t)$ for all choices of $y(\cdot)$. The proof is thus a "suped-up" version of the proof of Lemma 4.1. The points y_j have been chosen so that no matter what $y(s)$ is, it will be close to some y_j.

The numerator on the right side of (6.4) is easy to analyze. Of course,

$$\int_0^t P[G_s]\,ds\geqslant\int_{t/2}^t P[G_s]\,ds. \tag{6.5}$$

For given s, choose j so that $y(s) \in D_{\sqrt{t},j}$. Since $y_j \in D_{\sqrt{t},j}$,

$$\|y_j - y(s)\| \leqslant 2\sqrt{t}.$$

So by the local central limit theorem, if $s \geqslant t/2$, then

$$P[G_s] \geqslant P[Y_s^{y_j} = y(s)] \geqslant (c_2/t)^{d/2}, \tag{6.6}$$

for appropriate $c_2 > 0$ and t not too small. Plugging (6.6) into (6.5) gives

$$\int_0^t P[G_s]\, ds \geqslant c_3/t^{d/2-1}. \tag{6.7}$$

To analyze the denominator in (6.4), we introduce the following notation. Let J_t denote the smallest of the indices for which Y^{y_j} and $y(\cdot)$ meet by time t under $T \leqslant t$. Also, set

$$p_j = P[J_t = j \mid T \leqslant t].$$

The denominator can then be rewritten as

$$\sum_{j=1}^{\infty} p_j \int_0^t P[G_s \mid J_t = j]\, ds. \tag{6.8}$$

Decomposing the event G_s and noting that $Y_s^{y_j} = y(s)$ does not occur for $k < J_t$, we see that (6.8) is at most

$$\sum_{j=1}^{\infty} p_j \left[\int_0^t P[Y_s^{y_j} = y(s) \mid J_t = j]\, ds\right]$$
$$+ \sum_{j=1}^{\infty} p_j \left[\int_0^t P[Y_s^{y_k} = y(s),\ \text{some } k > j \mid J_t = j]\, ds\right]. \tag{6.9}$$

Since Y^{y_k} is independent of Y^{y_j} for $k > j$,

$$P[Y_s^{y_k} = y(s),\ \text{some } k > j \mid J_t = j] = P[Y_s^{y_k} = y(s), \text{some } k > j]$$
$$\leqslant P[Y_s^{y_k} = y(s),\ \text{some } k \geqslant 1] = P[G_s]. \tag{6.10}$$

So the second sum in (6.9) is

$$\leqslant \int_0^t P[G_s]\, ds. \tag{6.11}$$

Let T_j be the first time at which $Y_s^{y_j} = y(s)$. The integral in the first sum in (6.9) can be rewritten as

$$\int_0^t P[Y_s^{y_j} = y(s) \mid T_j \leqslant t]\, ds;$$

this is at most

$$\int_0^t P[Y^{y_j}_{s+T_j} = y(s+T_j) \mid T_j \leqslant t] \, ds. \tag{6.12}$$

The quantity in the last integral is the probability that a random walk starting at 0 is at $y(s+T_j) - y(T_j)$ at time s. Again applying the local central limit theorem, this quantity is at most

$$c_4/(s^{d/2} \vee 1). \tag{6.13}$$

Integrating (6.13), we see that the first sum in (6.9) is at most $h_d(t)$, where

$$\begin{aligned} h_d(t) &= c_5 \log t, \qquad d=2, \\ &= c_5, \qquad d \geqslant 3 \end{aligned} \tag{6.14}$$

(c_5 will depend on d).

Together, (6.11) and (6.14) show that

$$\int_0^t P[G_s \mid T \leqslant t] \, ds \leqslant h_d(t) + \int_0^t P[G_s] \, ds. \tag{6.15}$$

Using (6.4), we have that

$$P[T \leqslant t] \geqslant \left(h_d(t) \Big/ \int_0^t P[G_2] \, ds + 1 \right)^{-1}.$$

By (6.7) and (6.14), this is

$$\geqslant c_1 f_d(t),$$

for appropriate $c_1 > 0$. This implies (6.3). ∎

In Lemma 6.2 we apply Lemma 6.1 with the independent random walk Y^0_s (starting at the origin) being substituted for the path $y(s)$, and the number of random walks starting in each $D_{\sqrt{t}, j}$ being increased. The conclusion is that, except for an exponentially small probability, Y^0 hits a large number of these random walks by time t. The notation from Lemma 6.1 is used. We let $\mathscr{W}$ denote a set of independent random walks. Let $\mathfrak{L}(t)$ be the number of these random walks hit by Y^0 by time t. Also, let $E_R(t)$ denote the event for which $Y^0_s \in D_R$ for all $s \in [0, t]$. It follows from a simple large deviation estimate (using the moment generating function) and the reflection principle, that for appropriate $c_6, c'_6 > 0$,

$$\begin{aligned} P[E^c_R(t)] &\leqslant 4d \exp\{-c_6 R \log(1 + R/t)\} \\ &\leqslant 4d \exp\{-c'_6 (R \wedge (R^2/t))\} \end{aligned} \tag{6.16}$$

for $R, t>0$. We will use (6.16) later on (such as in Lemma 6.3 and Proposition 5). We note here that plugging in $R=t^{\delta}$, $\delta>1/2$, one gets

$$P[E^c_{t^\delta}(t)] \leqslant 4d \exp\{-c'_6 t^{\delta'}\}, \tag{6.17}$$

where $\delta'=\delta \wedge (2\delta-1)$.

Lemma 6.2. Assume that there are at least $[\alpha t^{d/2}]+1$ random walks in $\mathscr{W}$ initially contained in each cube $D_{\sqrt{t},j}$ intersecting D_{t^δ}, with $\alpha>0$, $\delta>1/2$. Then, for large enough t (not depending on the initial positions),

$$P[\mathfrak{L}(t) \leqslant c_1 \alpha g_d(t)/2;\ E_{t^\delta}(\mathrm{t})] \leqslant \exp\{-\beta c_1 \alpha g_d(t)/4\} \tag{6.18}$$

where β is as in Lemma 4.3 and c_1 as in Lemma 6.1.

Proof. By assumption, we can construct subsets $\mathscr{W}^1,..., \mathscr{W}^n$, $n=[\alpha t^{d/2}]+1$, of $\mathscr{W}$ which contain distinct random walks starting in $D_{\sqrt{t},j}$ for each $D_{\sqrt{t},j}$ intersecting D_{t^δ}. Let T^k be the first time at which Y^0 hits a member of $\mathscr{W}^k$. On $E_{t^\delta}(t)$ the hypothesis of Lemma 6.1 is satisfied with $y(s)=Y^0_s$ for each k. Therefore for each k,

$$P[T^k \leqslant t \mid Y^0] \geqslant c_1 f_d(t), \tag{6.19}$$

on $E_{t^\delta}(t)$. The events $\{T^k \leqslant t\}$ are independent given Y^0. So we can apply the corollary of Lemma 4.3 to (6.19) to conclude that

$$P[\{\mathfrak{L}(t) \leqslant c_1 n f_d(t)/2\} \mid Y^0] \leqslant \exp\{-\beta c_1 n f_d(t)/4\} \tag{6.20}$$

on $E_{t^\delta}(t)$. Since the bound on the right does not depend on Y^0, (6.20) reduces to

$$P[\mathfrak{L}(t) \leqslant c_1 \alpha g_d(t)/2;\ E_{t^\delta}(t)] \leqslant \exp\{-\beta c_1 \alpha g_d(t)/4\}. \quad \blacksquare$$

On account of Lemma 5.1, it is not difficult to see that the hypothesis of Lemma 6.2 will be satisfied for the B particles of the process ξ at any given time t, t not too small, except on an event of probability $\exp\{-\alpha' t^{d/2}\}$, $\alpha'>0$. More explicitly, one can choose $\alpha=(r_B-r_A)/2$ in Lemma 6.2 and $\alpha'=(r_B-r_A)^2/24r_B$. For our purposes, we may omit this small exceptional event. The conclusion of Lemma 6.2 together with (6.17) is that a random walk typically hits on the order of $\alpha g_d(t)$ random walks which correspond to B particles, but do not disappear. The probability of the exceptional event is again small, being bounded by exponentials with exponents of the form $-C\alpha g_d(t)$ (the right order) and $-c_6 t^{\delta'}$ (where ultimately, we will choose $\delta' \geqslant 1$). So applying Lemma 6.2 to ξ_t, where Y^0_s, $s \in [0, t]$, corresponds to the motion of an A particle over $[t, 2t]$, we see that this A particle will disappear by time $2t$ if at least one of these approximately $\alpha g_d(t)$ B-particles from time t still exists by time $2t$. This reasoning will ultimately produce Proposition 5 for $d \geqslant 2$. It is however conceivable that all of these B particles unluckily meet other A particles before having a chance to meet this particular one. The purpose of Lemmas 6.3–6.6 is to show that with high probability there are fewer than $\alpha g_d(t)$ A-particles which pass through D_{2t^δ}, $\delta > 1/2$, over $[t, 2t]$. Most A particles will have already disappeared by then. (The case $d=2$ will actually require some further work.) It will therefore be almost impossible for all these B particles (which must visit D_{t^δ} and hence usually remain in D_{2t^δ}) to avoid hitting our specified A particle.

Some Properties of ${}_t\tilde{\xi}$—Lemmas 6.3 and 6.4

For Lemmas 6.5 and 6.6 we will be interested in eliminating the effect in D_{2t^δ} of A particles from ξ_s which are ever outside D_{3t^δ}. We therefore introduce the process ${}_t\tilde{\xi}_s$ consisting of A and B particles which execute random walks and annihilate each other as before, but for which A particles disappear upon reaching $D^c_{3t^\delta}$. By defining ${}_t\tilde{\xi}$ on the same percolation substructure $\mathscr{P}$ as ξ, one obtains a natural coupling between the two processes. To compare ξ and ${}_t\tilde{\xi}$, one can proceed along the lines of (3.2) and Lemma 3.1. At each $y \in D^c_{3t^\delta}$ for which $\xi_{0-}(y) \neq {}_t\tilde{\xi}_{0-}(y)$, one introduces random walks $X^{y,i}_s$, $i = 1, 2,..., I_y$, which evolve according to $\mathscr{P}$ as specified between (3.1) and (3.2) (the number I_y at y being the difference ${}_t\tilde{\xi}_{0-}(y) - \xi_{0-}(y) =$ number of A particles initially at y for ξ). Also, introduce random walks $X^{\sigma_k}_s$, $s \geqslant \sigma_k$, $k = 1, 2,...$, which evolve in the same manner and are created when an A particle starting from D_{3t^δ} first moves outside at time σ_k. (This last part can be avoided with a little more work in Lemma 6.6.) The number of these random walks at a site x at time s equals ${}_t\tilde{\xi}_s(x) - \xi_s(x)$. The random walks evolve independently; since they all originate from "extra A particles," they will not annihilate one another. If one wishes, one can construct $X^{y,i}$, X^{σ_k} by inductively applying the procedure between (3.1) and (3.2).

We let ${}_t\mathscr{C}$ denote the set of the above random walks $X^{y,i}$, X^{σ_k} which are ever in D_{2t^δ} up to time $2t$. $|{}_t\mathscr{C}|$ will be used in Lemma 6.6 to bound the difference in the number of A particles visiting D_{2t^δ} up to time $2t$ for the processes ξ and ${}_t\tilde{\xi}$. Lemma 6.3 shows that $|{}_t\mathscr{C}|$ is typically small. The basic idea is to show using (6.16) that with high probability, only a few of the random walks $X^{y,i}$, X^{σ_k} can cross from ∂D_{3t^δ} to D_{2t^δ}, $\delta > 1/2$, by time $2t$.

Lemma 6.3. Assume that A and B particles are initially distributed over

$\mathbb{Z}^d$ according to (1.1) with $r_A < r_B$. For $\delta > 1/2$ and any constant $c_7 > 0$, there exists $c_8 > 0$ so that for large t,

$$P[|_t\mathscr{C}| \geqslant c_7(r_B - r_A)\, g_d(t)] \leqslant \exp\{-c_8(r_B - r_A)\, g_d(t)\}. \tag{6.21}$$

Also, for appropriate $c_9 > 0$, and large t,

$$P[_t\mathscr{C} \neq \varnothing] \leqslant \exp\{-c_9 t^{\delta'}\}, \tag{6.22}$$

where $\delta' = \delta \wedge (2\delta - 1)$.

Note that the bound in (6.1) is not sharp, although it suffices for Lemma 6.6. In particular, (6.22) implies (6.21) for $\delta > 1$.

Proof of Lemma 6.3. We define a family $\{Y^{y,i}\}$, $y \in \mathbb{Z}^d$, of independent random walks starting from ξ^A_{0-} as follows. For $y \in D^c_{3t^\delta}$, $1 \leqslant i \leqslant I_y$, set

$$Y^{y,i}_s = X^{y,i}_s \qquad \text{for all } s.$$

For $y \in D_{3t^\delta}$, let I_y (as before) denote the number of A particles initially at y; let $Y^{y,i}_s$, $1 \leqslant i \leqslant I_y$, evolve according to the same percolation substructure $\mathscr{P}$ as ξ, but with no annihilation. If the corresponding A particle in ξ survives until it hits ∂D_{3t^δ} at time σ_k, set

$$Y^{y,t}_s = X^{\sigma_k}_s \qquad \text{for} \quad s \geqslant \sigma_k.$$

Clearly, the family of random walks $\{Y^{y,t}\}$ defined in this manner contains $\{X^{y,i}\} \cup \{X^{\sigma_k}\}$. Note that the family $\{Y^{y,i}\}$ has initial state ξ^A_{0-}, which is Poisson, and has density r_A at each point.

We consider the subset $_t\mathscr{A}$ of $\{Y^{y,i}\}$ where

$$\sup_{s \in [0,2t]} \|Y^{y,i}_s - y\| \geqslant t^\delta/2. \tag{6.23}$$

Note that for either $X^{y,i}$ or X^{σ_k} to enter D_{2t^δ}, (6.23) must hold. By (6.16), for all y, i,

$$\alpha_t \overset{\text{def}}{=} P[\sup_{s \in [0,2t]} \|Y^{y,t}_s - y\| \geqslant t^\delta/2] \leqslant 4d \exp\{-c'_6 t^{\delta'}/8\},$$

where $\delta' = \delta \wedge (2\delta - 1)$.

Denote by $_t\eta_s$ the process counting the number of random walks in $_t\mathscr{A}$ at time s at each point in $\mathbb{Z}^d$. We are interested in $_t\eta_{2t}$. Since the random walks $Y^{y,i}$ evolve independently, $_t\eta_{2t}$ is a Poisson random measure. By the spatial homogeneity of ξ^A_{0-} and $_t\mathscr{A}$, $_t\eta_{2t}$ has density

$$r_A \alpha_t \leqslant 4dr_A \exp\{-c'_6 t^{\delta'}/8\}.$$

Set $R = \frac{5}{2}t^{\delta}$. $\mathfrak{D}_R(2t; {}_t\eta)$ is therefore Poisson with mean

$$\leqslant (\tfrac{5}{2}t^{\delta})^d 4dr_A \exp\{-c_6' t^{\delta'}/8\} \leqslant \tfrac{2}{3}c_{10}\, g_d(t)$$

for any $c_{10} > 0$ and large t. We use Lemma 5.1 in the simplest case where $t = 0$ and the minority type does not exist. (Or, one can just reapply the moment generating functions there.) It follows that

$$P[\mathfrak{D}_R(2t; {}_t\eta) \geqslant c_{10}\, g_d(t)] \leqslant \exp\{-c_{10}\, g_d(t)/36\} \tag{6.24}$$

for large t.

We want the analog of (6.24), but for all $s \in [0, 2t]$ rather than just $2t$. We get this by standard reasoning using the first time the random walk $Y^{y,i}$ enters $D_{2t^{\delta}}$ after crossing $\partial D_{3t^{\delta}}$. Let ${}_t\check{\mathscr{C}}$ denote the set of the (at most) $4c_{10}\, g_d(t)$ first $Y^{y,i}$ to thus enter $D_{2t^{\delta}}$ by time $2t$. Note that

$$|{}_t\mathscr{C}| \wedge [4c_{10}\, g_d(t)] \leqslant |{}_t\check{\mathscr{C}}|, \qquad {}_t\check{\mathscr{C}} \subset {}_t\mathscr{A}. \tag{6.25}$$

By the strong Markov property, for each (y, i),

$$P[Y_{2t}^{y,i} \in D_R \mid (y, i) \in {}_t\check{\mathscr{C}}] \geqslant P[Y_s^0 \in D_{t^{\delta}/2} \text{ for all } s \in [0, 2t]],$$

where Y^0 denotes a random walk starting at 0. Since $\delta > 1/2$, the right side can be chosen as close to 1 as desired by choosing t large. So

$$E[\#(Y_{2t}^{y,i} \in D_R, (y, i) \in {}_t\check{\mathscr{C}})] \geqslant 2c_{10}\, g_d(t)\, P[|{}_t\check{\mathscr{C}}| = [4c_{10}\, g_d(t)]].$$

The quantity in the expectation is of course bounded by $4c_{10}\, g_d(t)$. It is therefore easy to check that

$$P[\#(Y_{2t}^{y,i} \in D_R, (y, i) \in {}_t\check{\mathscr{C}}) \geqslant c_{10}\, g_d(t)] \geqslant \tfrac{1}{4}P[|{}_t\check{\mathscr{C}}| = [4c_{10}\, g_d(t)]].$$

Combining this with (6.25), we see that

$$\begin{aligned} P[|{}_t\mathscr{C}| \geqslant 4c_{10}\, g_d(t)] &\leqslant P[|{}_t\check{\mathscr{C}}| = [4c_{10}\, g_d(t)]] \\ &\leqslant 4P[\#(Y_{2t}^{y,i} \in D_R, (y, i) \in {}_t\check{\mathscr{C}}) \geqslant c_{10}\, g_d(t)]. \end{aligned}$$

By (6.24) and (6.25), this is

$$\begin{aligned} &\leqslant 4P[\mathfrak{D}_R(2t; {}_t\eta) \geqslant c_{10}\, g_d(t)] \\ &\leqslant \exp\{-c_{10}\, g_d(t)/36\}. \end{aligned}$$

If we plug in $c_7 = 4c_{10}/(r_B - r_A)$ and $c_8 = c_{10}/36(r_B - r_A) = c_7/144$, we get

$$P[|{}_t\mathscr{C}| \geqslant c_7(r_B - r_A)\, g_d(t)] \leqslant \exp\{-c_8(r_B - r_A)\, g_d(t)\},$$

which is (6.21).

The reasoning for (6.22) is simpler. As before, a random walk entering D_{2t^δ} before time $2t$ will typically still be in D_R, $R=\frac{5}{2}t^\delta$, at $2t$. One obtains from this and (6.25) that

$$P[{}_t\mathscr{C}\neq\varnothing]\leqslant P[{}_t\check{\mathscr{C}}\neq\varnothing]\leqslant 2P[\mathfrak{D}_R(2t;{}_t\eta)\neq 0]\leqslant E[\mathfrak{D}_R(2t;{}_t\eta)]=R^d r_A\alpha_t.$$

As before, the last quantity is

$$\leqslant 4dr_A R^d\exp\{-c_6' t^{\delta'}/8\}.$$

For $c_9<c_6'/8$ and t large enough, this is

$$\leqslant\exp\{-c_9 t^{\delta'}\},$$

which gives us (6.22). ∎

In Lemma 6.6 and Proposition 5, we will iterate the corollary of Lemma 6.5, which makes basic assumptions on the initial densities of A and B particles in the cubes $D_{R,j}$, $j\in\mathbb{Z}$, the disjoint translations of D_R (introduced before Lemma 6.1). Here, $\mathscr{J}^\delta$, $\delta>1/2$, will denote those indices j with

$$D_{R,j}\cap D_{3t^\delta}\neq\varnothing.$$

In Lemma 6.4, we show that $\mathfrak{D}^B_{R,j}(s;{}_t\tilde{\xi})$ is typically large and $\mathfrak{D}^A_{R,j}(s;{}_t\tilde{\xi})$ typically small for all $j\in\mathscr{J}^\delta$; ${}_t\tilde{\xi}$ is the process introduced earlier, with A particles killed when they reach $D^c_{3t^\delta}$. Here, we set

$$\begin{aligned} s_k&=kLt/\log t, && d=2,\\ &=kLt^{2/d}, && d\geqslant 3,\end{aligned}\tag{6.26}$$

where $L>0$ will be chosen later (and will be large); $k=0,\dots,K$, with $K=[(\log t)/L]$ in $d=2$ and $K=[t^{(d-2)/d}/L]$ in $d\geqslant 3$. In both cases $s_K\sim t$. (One can replace the exponent $2/d$ in $d\geqslant 3$ with any choice in $[2/d,1]$.) Set $R_1=\sqrt{s_1}$. The lemma is a simple consequence of Lemma 5.1.

Lemma 6.4. Assume that A and B particles are initially distributed according to (1.1) with $r_A<r_B$, but where the A particles are restricted to D_{3t^δ}. Then for large t and any $N>0$,

$$\begin{aligned} &P[\mathfrak{D}^B_{R_1,j}(s_k;{}_t\tilde{\xi})\leqslant\tfrac{1}{2}(r_B-r_A)R_1^d \text{ for some } k\leqslant K, j\in\mathscr{J}^\delta]\\ &\qquad\leqslant\exp\{-Ng_d(t)\},\end{aligned}\tag{6.27}$$

and

$$\begin{aligned} &P[\mathfrak{D}^A_{R_1,j}(s_k;{}_t\tilde{\xi})\geqslant\tfrac{3}{2}r_A R_1^d \text{ for some } k\leqslant K, j\in\mathscr{J}^\delta]\\ &\qquad\leqslant\exp\{-Ng_d(t)\},\end{aligned}\tag{6.28}$$

for L chosen large enough (which depends on r_A, r_B but not on δ).

Proof. In the natural coupling between ξ and ${}_t\tilde{\xi}$, $\xi_s^B \leqslant {}_t\tilde{\xi}_s^B$ for all s. So the left side of (6.27) is at most

$$P[\mathfrak{D}_{R_1,j}^B(s_k;\xi) \leqslant \tfrac{1}{2}(r_B - r_A) R_1^d \text{ for some } k \leqslant K, j \in \mathscr{J}^\delta],$$

where ξ has the initial measure given in (1.1). By Lemma 5.1,

$$P[\mathfrak{D}_{R_1,j}^B(s_k;\xi) \leqslant \tfrac{1}{2}(r_B - r_A) R_1^d] \leqslant \exp\left\{\frac{-(r_B - r_A)^2 R_1^d}{24 r_B}\right\} \tag{6.29}$$

for each k, j. Summing these probabilities gives

$$\begin{aligned} &P[\mathfrak{D}_{R_1,j}^B(s_k;\xi) \leqslant \tfrac{1}{2}(r_B - r_A) R_1^d \text{ for some } k \leqslant K, j \in \mathscr{J}^\delta] \\ &\quad \leqslant \frac{3^d}{L} t^{\delta d + 1} \exp\left\{\frac{-(r_B - r_A)^2 R_1^d}{24 r_B}\right\}. \end{aligned}$$

For large enough choice of L in (6.26) and large t, this is at most $\exp\{-N g_d(t)\}$. (This is the reason for our choice of s_1, R_1.) This shows (6.27). For (6.28), one can apply Lemma 5.1 again, this time substituting r_A for r_B and 0 for r_A in the lemma; alternatively, one can use the moment generating function to directly show the analog of (6.29),

$$P[\mathfrak{D}_{R_1,j}^A(s_k;\xi) \geqslant \tfrac{3}{2} r_A R_1^d] \leqslant \exp\left\{\frac{-(r_B - r_A)^2 R_1^d}{24 r_B}\right\}$$

for each k, j. One continues with the same reasoning as for (6.27). ∎

A Preliminary Iteration Scheme—Lemmas 6.5 and 6.6

Lemma 6.5, its corollary, and Lemma 6.6 present the machinery by which the density of A particles is shown to decrease sufficiently as time increases so as to allow application of Lemma 6.2. The basic idea is that given sufficiently many B particles in each cube $D_{R_1,j}$, a fixed proportion of A particles starting in D_{3t^δ} will have hit B particles by time $s_1 = R_1^2$. The probability that this occurs will be very high if there are not too few A particles (or too terribly many) and they are not bunched up in just a few cubes $D_{R_1,j}$. Iteration of this procedure gives a sufficiently rapid decrease of A particles to apply Lemma 6.2. (In $d = 2$, one needs to work a little harder.)

In Lemma 6.5, we carry out the first part of this argument. We find it convenient to work with the process $\hat{\eta}$ consisting of A and B particles undergoing independent random walks which do not interact, and has initial state given by (1.1) but where A particles are initially restricted to

D_{3t^δ} (no killing of A particles is assumed). Let $\mathcal{M}_{3t^\delta}(s)$ denote the number of sites in D_{3t^δ} with at least one particle of each type at time s. Lemma 6.5 says that under the right conditions, $\mathcal{M}_{3t^\delta}(s)$ will typically be large. If $\mathcal{M}_{3t^\delta}(s)$ is large, then a given proportion of the A particles in ${}_t\bar{\xi}$ must disappear by time s; this is utilized in Corollary 1. The result one obtains is iterated in Lemma 6.6.

Lemma 6.5. Let $R=\sqrt{s}\geqslant t^\varepsilon$ for some $\varepsilon>0$. Assume that for some $C_1, C_2, M>0$,

$$\mathfrak{D}^B_{R,j}(0;\hat{\eta})\geqslant C_1R^d \quad \text{for all } j\in\mathcal{J}^\delta, \tag{6.30}$$

$$\mathfrak{D}^A_{R,j}(0;\hat{\eta})\leqslant C_2R^d \quad \text{for all } j\in\mathcal{J}^\delta, \tag{6.31}$$

and

$$\mathfrak{D}^A_{3t^\delta}(0;\hat{\eta})=\sum_{j\in\mathcal{J}^\delta}\mathfrak{D}^A_{R,j}(0;\hat{\eta})\geqslant M. \tag{6.32}$$

Then for appropriate $C_3, C_4>0$ depending on C_1, C_2, but not δ,

$$P[\mathcal{M}_{3t^\delta}(s)\leqslant C_3M]\leqslant\exp\{-C_4(M\wedge R^d)\} \tag{6.33}$$

for large t.

Proof. The basic idea is as follows. Let Y^y_s denote a random walk with $Y^y_0=y$, and set

$$h_s=\min_{x,\,y\in D_R}P[Y^y_s=x]. \tag{6.34}$$

Note that for s not too small, it follows from the local central limit theorem that

$$h_s\geqslant C_5/R^d \tag{6.35}$$

for appropriate $C_5>0$. Consequently, if $y\in D_R$ and $\mathcal{Y}\subset D_R$ with $|\mathcal{Y}|\geqslant C_6R^d$, then

$$P[Y^y_s\in\mathcal{Y}]\geqslant C_5C_6. \tag{6.36}$$

Using (6.36) together with (6.30), we will show that for all $j\in\mathcal{J}^\delta$, at least a fixed proportion of $D_{R,j}$ is at time s covered by B particles with high probability. Using (6.36) again but with (6.31) and (6.32), we will show that each individual A particle will with good probability be at one of these sites, and that these particles are not concentrated at too few such sites. (6.33) will follow from the large deviations estimates in Lemma 4.3.

Let $y_{i,j}$, $i=1,..., n$, $j \in \mathscr{J}^{\delta}$, be an ordering of the positions of the "first"

$$n=[(C_1 \wedge \tfrac{1}{2}) R^d] \tag{6.37}$$

B-particles in $D_{R,j}$ at time 0. Denote by $Y^{y_{i,j}}$ the random walks associated with these particles. For each j, we define the events $G_{i,j}$ inductively as follows. Let

$$\mathscr{Y}^{i,j}=\{Y_s^{y_{1,j}},..., Y_s^{y_{i,j}}\} \cap D_{R,j}; \tag{6.38}$$

abbreviate for $i=n$ by setting $\mathscr{Y}^j=\mathscr{Y}^{n,j}$. Also, let

$$\hat{\mathscr{Y}}^{i,j}=D_{R,j}-\mathscr{Y}^{i,j}. \tag{6.39}$$

We set

$$G_{i,j}=\{\omega: Y_s^{y_{i,j}} \in \hat{\mathscr{Y}}^{i-1,j}\}. \tag{6.40}$$

That is, $G_{i,j}$ occurs when $Y_s^{i,j}$ is at a site in $D_{R,j}$ occupied by no "previous" B particle.

Now, on account of (6.37),

$$|\hat{\mathscr{Y}}^{i-1,j}| \geqslant \tfrac{1}{2} R^d \tag{6.41}$$

for all i, j, no matter what the behavior of $Y_s^{y_{i',j}}$, $i'<i$. So by (6.36),

$$P[G_{i,j} \mid \sigma(G_{1,j},..., G_{i-1,j})] \geqslant C_5/2 \tag{6.42}$$

always holds. It is therefore not difficult to see that for each j, the n-tuple $1_{G_{1,j}},..., 1_{G_{n,j}}$ dominates i.i.d. random variables ${}^B W_1,..., {}^B W_n$ with

$$P[{}^B W_i=1]=C_5/2, \qquad P[{}^B W_i=0]=1-C_5/2. \tag{6.43}$$

(That is, the n-tuples can be coupled so that ${}^B W_i=1$ implies $G_{i,j}$ occurs.) Applying Corollary 1 of Lemma 4.3, one obtains

$$P\left[\sum_{i=1}^n 1_{G_{i,j}} \leqslant C_5 n/4\right] \leqslant P\left[\sum_{i=1}^n {}^B W_i \leqslant C_5 n/4\right] \leqslant e^{-C_5 \beta n/4} \tag{6.44}$$

for all j, where $\beta>0$ is as in the corollary. Let G be the event

$$\begin{aligned} G &= \left\{\omega: \sum_{i=1}^n 1_{G_{i,j}} > C_5 n/4 \text{ for all } j \in \mathscr{J}^{\delta}\right\} \\ &= \{\omega: |\mathscr{Y}^j| > C_5 n/4 \text{ for all } j \in \mathscr{J}^{\delta}\}. \end{aligned} \tag{6.45}$$

On account of (6.44),

$$P[G^c] \leqslant 3^d t^{\delta d} e^{-C_5 \beta n/4} \leqslant e^{-C_7 n} \tag{6.46}$$

for appropriate $C_7 > 0$ and large enough t, since $n \geqslant \text{const. } t^{\varepsilon d}$.

We next concern ourselves with the motion of the A particles. Let $z_{i,j}$ and $Z^{z_{i,j}}$ be the initial positions and random walks associated with the A particles, with $j \in \mathscr{J}^\delta$. We let $i = 1,..., I_j$, where

$$I_j = [C_5 n/8] \wedge \mathfrak{D}^A_{R,j}(0; \hat{\eta}). \tag{6.47}$$

On account of (6.31) and (6.37),

$$\frac{C_5 n/8}{\mathfrak{D}^A_{R,j}(0; \hat{\eta})} \geqslant \frac{C_5(C_1 \wedge \frac{1}{2})}{8C_2}.$$

Set $C_8 = (C_5(C_1 \wedge \frac{1}{2})/8C_2) \wedge 1$. Together with (6.32), this implies

$$\sum_{j \in \mathscr{J}^\delta} I_j \geqslant C_8 M.$$

We can for convenience choose $I'_j \leqslant I_j$ for all j with

$$m \equiv \sum_{j \in \mathscr{J}^\delta} I'_j = [C_8 M]. \tag{6.48}$$

We proceed to construct events $H_{i,j}$, $i = 1,..., m$, $j \in \mathscr{J}^\delta$, associated with $Z^{z_{i,j}}$, in a manner similar to the construction of $G_{i,j}$ from $Y^{y_{i,j}}$. First let

$$\mathscr{Z}^{i,j} = \{Z_s^{z_{1,j}},..., Z_s^{z_{i,j}}\} \cap D_{R,j}. \tag{6.49}$$

Also, let

$$\hat{\mathscr{Z}}^{i,j} = (D_{R,j} \cap \mathscr{Y}^j) - \mathscr{Z}^{i-1,j}. \tag{6.50}$$

We set

$$H_{i,j} = \{\omega : Z_s^{z_{i,j}} \in \hat{\mathscr{Z}}^{i-1,j}\}. \tag{6.51}$$

That is, $H_{i,j}$ occurs when $Z_s^{i,j}$ is at a site in $D_{R,j}$ also occupied by one of the specified B particles, but by no "previous" A particle.

We mimic the reasoning of (6.41)–(6.46). By (6.45) and (6.47),

$$|\hat{\mathscr{Z}}^{i-1,j}| \geqslant [C_5 n/8] \quad \text{on} \quad G \tag{6.52}$$

for all i, j, no matter what the behavior of $Z_s^{z_{i',j}}$, $i' < i$, or $Z_s^{z_{i',j'}}$, $\forall i'$, $j' \neq j$.

Let $\mathscr{H}_{i,j}$ be the σ-algebra generated by G, $H_{i',j}$ with $i' < i$, and by $H_{i',j'}$ with $j' \neq j$. Because of (6.37), (6.52), and (6.36), on G,

$$P[H_{i,j} \mid \mathscr{H}_{i,j}] \geqslant (C_5)^2(C_1 \wedge \tfrac{1}{2})/9 \geqslant C_9 \tag{6.53}$$

for R not too small, where $C_9 > 0$. As before, it is not difficult to see that the m-tuple $1_{H_{i,j}}$, $i = 1,..., I_j'$, $j \in \mathscr{J}^{\delta}$, dominates i.i.d. random variables ${}^AW_1,..., {}^AW_m$, on G, with

$$P[{}^AW_i = 1] = C_9, \qquad P[{}^AW_i = 0] = 1 - C_9, \tag{6.54}$$

where m is given in (6.48). Applying Corollary 1 of Lemma 4.3 again, one obtains

$$P\left[\sum_{i,j} 1_{H_{i,j}} \leqslant C_9 m/2 \mid G\right] \leqslant P\left[\sum_{i=1}^{m} {}^AW_i \leqslant C_9 m/2\right] \leqslant e^{-C_{10}m} \tag{6.55}$$

for appropriate $C_{10} > 0$. Plugging (6.48), (6.46), and then (6.37) into (6.55), it follows that

$$P\left[\sum_{i,j} 1_{H_{i,j}} \leqslant C_3 M\right] \leqslant e^{-C_{11}M} + e^{-C_7 n} \leqslant \exp\{-C_4(M \wedge R^d)\}, \tag{6.56}$$

for appropriate choices of C_3, C_4, $C_{11} > 0$. Of course,

$$\mathscr{M}_{3t^{\delta}}(s) \geqslant \sum_{i,j} 1_{H_{i,j}},$$

and so (6.33) follows from (6.56). ∎

Lemma 6.5 has the following consequence when reinterpreted in terms of the process ${}_t\tilde{\xi}$.

Corollary 1. Let $R = \sqrt{s} \geqslant t^{\varepsilon}$ for some $\varepsilon > 0$. Assume that for some $C_1, C_2, l > 0$,

$$\mathfrak{D}^B_{R,j}(0; {}_t\tilde{\xi}) \geqslant C_1 R^d \qquad \text{for all} \quad j \in \mathscr{J}^{\delta}, \tag{6.57}$$

$$\mathfrak{D}^A_{R,j}(0; {}_t\tilde{\xi}) \leqslant C_2 R^d \qquad \text{for all} \quad j \in \mathscr{J}^{\delta}, \tag{6.58}$$

and

$$\mathfrak{D}^A_{3t^{\delta}}(0; {}_t\tilde{\xi}) \geqslant l g_d(t). \tag{6.59}$$

Then for C_3, $C_4 > 0$ as given in Lemma 6.5,

$$P\left[\frac{\mathfrak{D}^A_{3t^{\delta}}(s; {}_t\tilde{\xi})}{\mathfrak{D}^A_{3t^{\delta}}(0; {}_t\tilde{\xi})} \geqslant 1 - \frac{C_3}{2}\right] \leqslant \exp\{-C_4((l g_d(t)) \wedge R^d)\} \tag{6.60}$$

for large t.

Proof. The hypotheses of Lemma 6.5 are satisfied with

$$M = \mathfrak{D}^A_{3t^\delta}(0; \hat{\eta}) = \mathfrak{D}^A_{3t^\delta}(0; {}_t\bar{\xi}) \geq l g_d(t).$$

So by (6.33), for large t

$$P[\mathscr{M}_{3t^\delta}(s) \leq C_3 M] \leq \exp\{-C_4(M \wedge R^d)\}$$
$$\leq \exp\{-C_4((l g_d(t)) \wedge R^d)\} \qquad (6.61)$$

for the process $\hat{\eta}$. Under the event

$$\{\mathscr{M}_{3t^\delta}(s) > C_3 M\}, \qquad (6.62)$$

at least $C_3 M$ pairs of A and B particles cohabit the same sites at time s. For ${}_t\bar{\xi}$ (which has the same percolation substructure and initial data as $\hat{\eta}$), this means that either at least $C_3 M/2$ A-particles or at least $C_3 M/2$ B-particles have been annihilated by time s, since A and B particles cannot be present at the same site. For each B particle which disappears, there corresponds an A particle which disappears. (The converse is false for ${}_t\bar{\xi}$.) So in either case, at least $C_3 M/2$ A-particles are lost. It follows that under (6.62),

$$\frac{\mathfrak{D}^A_{3t^\delta}(s; {}_t\bar{\xi})}{\mathfrak{D}^A_{3t^\delta}(0; {}_t\bar{\xi})} \geq 1 - \frac{C_3}{2}.$$

(6.60) follows from this and (6.61). ∎

Let $\mathfrak{N}^A_R(t)$ denote the number of A particles which intersect D_R at any time $s \in [t, 2t]$ for the process ξ. In Lemma 6.6, we show that $\mathfrak{N}^A_{2t^\delta}(t)$ is typically at most of order $g_d(t)$. This will be combined with Lemma 6.2 in Proposition 5. Lemma 6.6 is shown by iterating Corollary 1 of Lemma 6.5, and then using Lemma 6.3 to go from ${}_t\bar{\xi}$ to ξ. Lemma 6.4 is used to verify that the hypotheses of Lemma 6.5 are typically satisfied here. We will use the following notation. We introduce the events

$$H^B_k = \{\mathfrak{D}^B_{R_1,j}(s_i; {}_t\bar{\xi}) > \tfrac{1}{2}(r_B - r_A) R_1^d \text{ for all } i \leq k, j \in \mathscr{J}^\delta\}, \qquad (6.63)$$

$$H^A_k = \{\mathfrak{D}^A_{R_1,j}(s_i; {}_t\bar{\xi}) < \tfrac{3}{2} r_A R_1^d \text{ for all } i \leq k, j \in \mathscr{J}^\delta\}, \qquad (6.64)$$

$$H^g_k = \{\mathfrak{D}^A_{3t^\delta}(s_k; {}_t\bar{\xi}) \geq l g_d(t)\}, \qquad (6.65)$$

where s_k, R_1, $\mathscr{J}^\delta$ are defined at (6.26), and $l > 0$ (but small) will be chosen later. Set

$$H_k = H^A_k \cap H^B_k \cap H^g_k. \qquad (6.66)$$

Note that $H_k^g \downarrow$ as $k \uparrow$; therefore $H_k \downarrow$ as $k \uparrow$. Let

$$\kappa = \max\{k \in [0, K-1] : \omega \in H_k\}, \tag{6.67}$$

where K is defined below (6.26). Also, recall that ${}_t\xi_s$ is measurable with respect to the σ-algebra $\mathscr{F}_s$ (on the percolation substructure and initial state of ξ) introduced in Section 1. Of course, $H_k^A, H_k^B, H_k^g \in \mathscr{F}_{s_k}$.

Lemma 6.6. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$ according to (1.1) with $r_A < r_B$. For appropriate $\delta > 1/2$ (depending on r_A, r_B) and any constant $C_{12} > 0$, there exists $C_{13} > 0$ so that for large t,

$$P[\mathfrak{N}_{2t^\delta}^A(t) \geq C_{12}(r_B - r_A)\, g_d(t)] \leq \exp\{-C_{13}(r_B - r_A)\, g_d(t)\}. \tag{6.68}$$

For $d \geq 3$, one can choose $\delta > 1$.

Proof. The hypotheses of Corollary 1 of Lemma 6.5 are satisfied at time s_k on H_k, where $C_1 = \frac{1}{2}(r_B - r_A)$, $C_2 = \frac{3}{2} r_A$. So on H_k,

$$P\left[\frac{\mathfrak{D}_{3t^\delta}^A(s_{k+1};\, {}_t\xi)}{\mathfrak{D}_{3t^\delta}^A(s_k;\, {}_t\xi)} \geq 1 - \tfrac{1}{2}C_3 \,\Big|\, \mathscr{F}_{s_k}\right] \leq \exp\{-C_4((lg_d(t)) \wedge R_1^d)\}$$

(for large t). By assumption,

$$R_1^d = L^{d/2} g_d(t),$$

so the right side can be rewritten as

$$\exp\{-C_4(l \wedge L^{d/2})\, g_d(t)\}.$$

Choosing L large enough, this equals

$$\exp\{-C_4 l g_d(t)\}.$$

Iterating, we obtain that

$$P[\mathfrak{D}_{3t^\delta}^A(s_{\kappa+1};\, {}_t\xi) \geq (1 - \tfrac{1}{2}C_3)^\kappa\, \mathfrak{D}_{3t^\delta}^A(0;\, {}_t\xi)] \leq K \exp\{-C_4 l g_d(t)\}. \tag{6.69}$$

Note that $\mathfrak{D}_{3t^\delta}^A(s;\, {}_t\xi)$ is decreasing in s (under ${}_t\xi$, there are no A particles outside D_{3t^δ}), and that $s_k \leq t$. Since $s_0 = 0$,

$$\mathfrak{D}_{3t^\delta}^A(0;\, {}_t\xi) < 3^{d+1} r_A t^{\delta d} \quad \text{on} \quad H_K^A.$$

Also,

$$H_K^g \cap \{\kappa < K-1\} \subset (H_K^A)^c \cup (H_K^B)^c.$$

So (6.69) implies that

$$P[\mathfrak{D}^A_{3t^\delta}(t; {}_t\bar{\xi}) \geqslant ((1-\tfrac{1}{2}C_3)^K 3^{d+1} r_A t^{\delta d}) \vee (lg_d(t)); H^A_K \cap H^B_K] \leqslant K \exp\{-C_4 lg_d(t)\}. \tag{6.70}$$

In other words, if there are always enough B particles and few enough A particles to perform the iterations for all k, the iteration will only stop before K if $lg_d(t)$ has already been reached.

We apply Lemma 6.4 to estimate the right side of (6.70). Setting $N = C_4 l$ in the lemma, L can be chosen large enough so that

$$P[(H^A_K)^c \cup (H^B_K)^c] \leqslant 2\exp\{-C_4 lg_d(t)\}. \tag{6.71}$$

Together with (6.70), this implies that

$$P[\mathfrak{D}^A_{3t^\delta}(t; {}_t\bar{\xi}) \geqslant ((1-\tfrac{1}{2}C_3)^K 3^{d+1} r_A t^{\delta d}) \vee (lg_d(t))] \leqslant (K+2)\exp\{-C_4 lg_d(t)\}. \tag{6.72}$$

Now, K was chosen in (6.26) so that $K = [(\log t)/L]$ in $d=2$ and $K = [t^{(d-2)/d}/L]$ in $d \geqslant 3$. It is easy to check that for large t,

$$(1-\tfrac{1}{2}C_3)^K 3^{d+1} r_A t^{\delta d} < t^{1-\varepsilon'} \tag{6.73}$$

for some $\varepsilon' > 0$, if δ is chosen close enough to 1/2 in $d=2$; in $d \geqslant 3$ we can use almost anything, but content ourselves with $\delta > 1$. Note that the choice of δ in $d=2$ depends on C_3 and L, which in turn depend on r_A and r_B. For t not too small,

$$t^{1-\varepsilon'} \leqslant lg_d(t).$$

Using (6.73), (6.72) therefore simplifies to

$$P[\mathfrak{D}^A_{3t^\delta}(t; {}_t\bar{\xi}) \geqslant lg_d(t)] \leqslant (K+2)\exp\{-C_4 lg_d(t)\}. \tag{6.74}$$

Recall that ${}_t\bar{\xi}$ differs from ξ in that A particles automatically disappear upon entering $(D_{3t^\delta})^c$. (6.74) gives bounds on the probability that there are at least $lg_d(t)$ A-particles from ${}_t\bar{\xi}$ which visit $D_{2t^\delta} \subset D_{3t^\delta}$ after time t. Lemma 6.3, on the other hand, gives bounds on the probability that ξ and ${}_t\bar{\xi}$ differ by much on D_{2t^δ} and $s \in [0, 2t]$. The quantity ${}_t\mathscr{C}$ defined there is an upper bound on the difference in the number of A particles visiting D_{2t^δ} up to time $2t$ for ξ and ${}_t\xi$. By choosing $\delta > 1$ in $d \geqslant 3$, we conclude from (6.22) that for large t,

$$P[{}_t\mathscr{C} \neq \varnothing] \leqslant \exp\{-c_9 t^{\delta'}\}, \tag{6.75}$$

where $\delta' = \delta \wedge (2\delta - 1) > 1$. The right side of (6.75) is of smaller order than that in (6.74). Combining (6.74) and (6.75) gives

$$P[\mathfrak{N}^A_{2t^\delta}(t) \geqslant l g_d(t)] \leqslant (K+2) \exp\{-C_4 l g_d(t)\} + \exp\{-c_9 t^{\delta'}\}$$

for large t. Choosing $l = C_{12}(r_B - r_A)$ and $C_{13} < C_4 C_{12}$, we obtain (6.68) in $d \geqslant 3$. For $d = 2$ (and $d \geqslant 3$ if desired), use (6.21) to conclude that for any $c_7 > 0$ there is a $c_8 > 0$ so that

$$P[|{}_t\mathscr{C}| \geqslant c_7(r_B - r_A)\, g_d(t)] \leqslant \exp\{-c_8(r_B - r_A)\, g_d(t)\}.$$

Together with (6.74), this gives

$$P[\mathfrak{N}^A_{2t^\delta}(t) \geqslant (l + c_7(r_B - r_A))\, g_d(t)]$$
$$\leqslant (K+2) \exp\{-C_4 l g_d(t)\} + \exp\{-c_8(r_B - r_A)\, g_d(t)\}$$

for large t. Choosing $l = C_{12}(r_B - r_A)/2$, $c_7 = C_{12}/2$, and $C_{13} < (C_4 C_{12}/2) \wedge c_8$, we obtain (6.68) in $d = 2$ as well.

Sharp Upper Bounds on $\rho_A(t)$—Proposition 5

We can now use Lemmas 6.2 and 6.6 to prove Proposition 5. We will use the following notation. Define $\mathscr{J}^\delta_1$ similarly to $\mathscr{J}^\delta$ just above (6.26), with $\mathscr{J}^\delta_1$ denoting those indices j with

$$D_{\sqrt{t},j} \cap D_{t^\delta} \neq \varnothing.$$

Expand D_{t^δ} slightly to

$$\bar{D}_{t^\delta} = \bigcup_{j \in \mathscr{J}^\delta_1} D_{\sqrt{t},j};$$

define $\bar{\mathfrak{D}}_{t^\delta}$ analogously. Of course, $\bar{D}_{t^\delta} \subset D_{(4/3)t^\delta}$ for $\delta > 1/2$ and t large. Let η^t_s, $s \geqslant t$, denote the process consisting of A and B particles which execute noninteracting random walks and for which $\eta^t_t = \xi_t$. Let Y^0_s be a random walk with $Y^0_0 = 0$. (It may be thought of as corresponding to an A particle.) Set $\tilde{Y}^0_s = Y^0_s - Y^0_t$ for $s \in [t, 2t]$; then $\tilde{Y}^0_t = 0$. This "initial state" will simplify notation for us. Also, set $\tilde{\xi} = \xi - Y^0_t$. Since the movement of Y^0 is independent of ξ and ξ is translation invariant, $\tilde{\xi}$ has the same distribution as ξ. The same of course also holds for $\tilde{\eta}^t = \eta^t - Y^0_t$. We introduce the following quantities for the process $\tilde{\eta}^t$:

$$\mathfrak{L}^\delta(t) = \#\text{ of } B \text{ particles in } \bar{D}_{t^\delta} \text{ at time } t \text{ which meet } \tilde{Y}^0 \text{ over } [t, 2t]. \quad (6.76)$$

$$\mathfrak{L}^\delta_1(t) = \#\text{ of } B \text{ particles in (6.76) which remain in } D_{2t^\delta} \text{ over } [t, 2t]. \quad (6.77)$$

$$\mathfrak{L}^\delta_2(t) = \#\text{ of } B \text{ particles in (6.76) which do not remain in } D_{2t^\delta} \text{ over } [t, 2t]. \quad (6.78)$$

Of course, $\mathfrak{L}^\delta(t)=\mathfrak{L}_1^\delta(t)+\mathfrak{L}_2^\delta(t)$. Also, let $E^\delta(t)$ denote the event that $\tilde{Y}^0$ remains in D_{t^δ} over $[t, 2t]$. In $d=2$, we will also need to employ the analogs of (6.76)–(6.78) and $E^\delta(t)$, where time is translated t units (so $[t, 2t]$ becomes $[2t, 3t]$) but everything else remains the same. $\hat{Y}_s^0=Y_s^0-Y_{2t}^0$, $s\in[2t, 3t]$, will replace $\tilde{Y}_s^0$. To denote the analogs, we insert "$\wedge$," e.g., $\hat{\mathfrak{L}}^\delta(t)$, $\hat{E}^\delta(t)$. The same correspondence will be made for quantities introduced later. $\hat{\mathscr{K}}^\delta(t)$ will correspond to $\mathscr{K}^\delta(t)$, etc. As will be indicated, equivalent statements will hold for the "$\wedge$" quantities.

We summarize our basic reasoning. It will follow from Lemma 6.2 that on $E^\delta(t)$, $\mathfrak{L}^\delta(t)$ is typically of order $g_d(t)$. It is a simple large deviation estimate that $\mathfrak{L}_2^\delta(t)$ is small. So $\mathfrak{L}_1^\delta(t)$ must also be of order $g_d(t)$ ((6.80)–(6.82)). But on account of Lemma 6.6, these B particles remaining in D_{2t^δ} can only meet on the order of $g_d(t)$ A-particles. Choosing coefficients properly, $\mathfrak{L}_1^\delta(t)$ can be made larger than this last quantity. So even for the process $\tilde{\xi}$, at least one of these B particles will survive to hit $\tilde{Y}^0$ by time $2t$, except for a small probability ((6.83)–(6.85)). The probability of $\tilde{Y}^0$ leaving D_{t^δ} by time $2t$ is also not large ((6.86)). Associate $\tilde{Y}^0$ with the path of an A particle (with some continuation after the particle is annihilated). This reasoning therefore gives upper bounds for $\rho_A(2t)$. For $\delta>1$ $(d>2)$, the last probability ((6.86)) is small enough for our conclusion in Proposition 5 ((6.87)). For $\delta>1/2$ $(d=2)$, the estimate is not yet good enough. Our bound on the density of A particles is still too high. It gives a marked improvement however over the bound in Lemma 6.6. Using this improvement, one can repeat the above argument, this time with $\delta>1$. This gives the correct bounds in $d=2$ as well.

Proposition 5. Assume that A and B particles are initially distributed over $\mathbb{Z}^d$, $d\geqslant 2$, according to (1.1) with $r_A<r_B$. Then

$$\rho_A(t)\leqslant \exp\{-\lambda(r_B-r_A)\, g_d(t)\} \tag{6.79}$$

for appropriate $\lambda>0$ (depending on d) and large enough t.

Proof. We break the argument into two parts.

Part I—$d>2$ and preliminary bounds for $d=2$. For $d>2$, we will consider the evolution of a random walk Y_s^0 (with $Y_0^0=0$) over the time intervals $[0, t_1]$ and $[t_1, 2t_1]$, with $t_1=t/2$. For $d=2$, we will also use the interval $[2t_1, 3t_1]$ and set $t_1=t/3$. To demonstrate (6.79), it suffices to show that the probability that Y^0 does not meet any B-particles from ξ over $[t_1, 2t_1]$ (or over $[2t_1, 3t_1]$, in $d=2$) is at most as great as the probability given in (6.79). (A little thought shows that introducing Y^0 into the system instead of choosing an already present A particle in effect inserts another A particle at 0 into ξ. By Lemma 3.2, this will decrease the probability of an individual A particle being hit.) It will be convenient to use the quantities $\tilde{Y}^0$, $\tilde{\xi}$, and $\tilde{\eta}^{t_1}$ introduced just before (6.76). Recall that $\tilde{Y}^0_{t_1}=0$ and that $\tilde{\xi}$ and $\tilde{\eta}^{t_1}$ have the same distributions as ξ and η^{t_1} and are independent of $\tilde{Y}^0$. In the following computations, we will drop the superscript "$\sim$" from $\tilde{Y}$, $\tilde{\xi}$, and $\tilde{\eta}^{t_1}$ and write, e.g., $Y^0_{t_1}=0$.

We first note that by Lemma 5.1,

$$P[\mathfrak{D}^B_{\sqrt{t_1},j}(t_1;\eta^{t_1})\leqslant \tfrac{1}{3}(r_B-r_A)t_1^{d/2}+1 \text{ for some } j\in\mathscr{J}_1^\delta]$$
$$\leqslant t_1^{\delta d}\exp\left\{-\frac{(r_B-r_A)^2t_1^{d/2}}{24r_B}\right\}$$

for large t; we assume $\delta>1/2$. One can therefore apply Lemma 6.2 with $\alpha=(r_B-r_A)/3$ to obtain

$$P[\mathfrak{L}^\delta(t_1)\leqslant c_1(r_B-r_A)\,g_d(t_1)/6;E^\delta(t_1)]$$
$$\leqslant \exp\{-\beta\,c_1(r_B-r_A)\,g_d(t_1)/12\}+t_1^{\delta d}\exp\left\{-\frac{(r_B-r_A)^2t_1^{d/2}}{24r_B}\right\},$$

where the constants are the same as in the lemma. Simplifying, we see that for large t,

$$P[\mathfrak{L}^\delta(t_1)\leqslant C_{14}(r_B-r_A)\,g_d(t_1);E^\delta(t_1)]$$
$$\leqslant \exp\{-C_{15}(r_B-r_A)\,g_d(t_1)\},\tag{6.80}$$

where $C_{14}, C_{15}>0$ are appropriate constants.

On the other hand, by Lemma 5.1 (with $r_A=0$),

$$P[\bar{\mathfrak{D}}^B_{t_1^\delta}(t_1;\eta^{t_1})\geqslant 2r_Bt_1^{\delta d}]\leqslant \exp\{-r_Bt_1^{\delta d}/24\}$$

for large t. Also, by (6.16), the probability that an individual B particle from $\bar{D}_{t_1^\delta}$ leaves $D_{2t_1^\delta}$ by time t_1 is at most $4d\exp\{-c_6t_1^{\delta'}/4\}$, where $\delta'=\delta\wedge(2\delta-1)$. One can therefore apply the corollary of Lemma 4.3 to obtain

$$P[\mathfrak{L}_2^\delta(t_1)\geqslant C_{14}(r_B-r_A)\,g_d(t_1)/2]$$
$$\leqslant P[\#(B\text{ particles in }\bar{D}_{t_1^\delta}\text{ at }t_1,\text{ but exiting }D_{2t_1^\delta}\text{ over }[t_1,2t_1])$$
$$\geqslant C_{14}(r_B-r_A)\,g_d(t_1)/2]$$
$$\leqslant \exp\{-r_Bt_1^{\delta d}/24\}+\exp\{-\beta C_{14}(r_B-r_A)\,g_d(t_1)/4\}.$$

(We are setting $n=[2r_Bt_1^{\delta d}]$, $p=4d\exp\{-c_6t_1^{\delta'}/4\}$, and $\delta=$

$(C_{14}(r_B - r_A)\, g_d(t_1)/2np) - 1 \gg 1$ in the corollary.) Simplifying, we see that for large t,

$$P[\mathfrak{L}_2^\delta(t_1) \geqslant C_{14}(r_B - r_A)\, g_d(t_1)/2] \leqslant \exp\{-C_{16}(r_B - r_A)\, g_d(t_1)\} \tag{6.81}$$

since $\delta > 1/2$. Together, (6.80) and (6.81) show that

$$\begin{aligned} &P[\mathfrak{L}_1^\delta(t_1) \leqslant C_{14}(r_B - r_A)\, g_d(t_1)/2;\, E^\delta(t_1)] \\ &\quad \leqslant \exp\{-C_{17}(r_B - r_A)\, g_d(t_1)\}, \end{aligned} \tag{6.82}$$

where $C_{17} > 0$. (Note for use in $d = 2$ that the analog of (6.82) holds for $\hat{\mathfrak{L}}_1(t_1)$, since Lemma 6.2 can be applied at time $2t_1$ as well; the reasoning from (6.80) on is the same.)

We employ (6.82) together with Lemma 6.6. The lemma gives upper bounds on the number $\mathfrak{N}^A_{2t_1^\delta}(t_1)$ of A particles for the process ξ intersecting $D_{2t_1^\delta}$ over $[t_1, 2t_1]$. Choose $C_{12} = C_{14}/2$ there. It follows that

$$P[\mathfrak{N}^A_{2t_1^\delta}(t_1) \geqslant C_{14}(r_B - r_A)\, g_d(t_1)/2] \leqslant \exp\{-C_{13}(r_B - r_A)\, g_d(t_1)\} \tag{6.83}$$

for appropriate δ, with $\delta > 1$ in $d > 2$ and $\delta > 1/2$ in $d = 2$, and with $C_{13} > 0$. Fix this δ. (6.83) gives bounds on the number of B particles for the process ξ which remain in $D_{2t_1^\delta}$ until they are annihilated at some time in $[t_1, 2t_1]$ by an A particle; denote this number by $\mathfrak{N}^B_{2t_1^\delta}(t_1)$. Since only one B particle is annihilated by each A particle,

$$\mathfrak{N}^B_{2t_1^\delta}(t_1) \leqslant \mathfrak{N}^A_{2t_1^\delta}(t_1).$$

So by (6.83),

$$P[\mathfrak{N}^B_{2t_1^\delta} \geqslant C_{14}(r_B - r_A)\, g_d(t_1)/2] \leqslant \exp\{-C_{13}(r_B - r_A)\, g_d(t_1)\}. \tag{6.84}$$

Recall that ξ and η' are coupled together with

$$\xi_{t_1} = \eta^{t_1}_{t_1}; \qquad \xi^B_s \leqslant (\eta^{t_1})^B_s, \qquad s \geqslant t_1.$$

In particular, the B particles of ξ correspond exactly to the B particles of η^{t_1} except where the former have been annihilated by A particles. Let $\mathscr{K}^\delta(t_1)$ denote the number of B particles in ξ which meet Y^0 in $[t_1, 2t_1]$ (before the B particle is annihilated). It follows from (6.82) and (6.84) that for large t,

$$P[\mathscr{K}^\delta(t_1) = 0;\, E^\delta(t_1)] \leqslant \exp\{-C_{18}(r_B - r_A)\, g_d(t_1)\}, \tag{6.85}$$

where $C_{18} > 0$. That is, off of the exceptional events in (6.82) and (6.84), $\mathfrak{L}_1^\delta(t_1) > \mathfrak{N}^B_{2t_1^\delta}(t_1)$, in which case Y^0 meets at least one B particle in ξ during $[t_1, 2t_1]$.

We still need to look at $(E^{\delta}(t_1))^c$. On account of (6.17), for large t,

$$P[(E^{\delta}(t_1))^c] \leqslant 4d \exp\{-c_6' t_1^{\delta'}\}, \tag{6.86}$$

where $\delta' > 1$ in $d > 2$ and $\delta' > 0$ in $d = 2$. Together with (6.85), this shows that

$$P[\mathscr{K}^{\delta}(t_1) = 0] \leqslant \exp\{-C_{19}(r_B - r_A)\, g_d(t_1)\} \qquad \text{for} \quad d > 2, \tag{6.87}$$

where $C_{19} > 0$. Multiplication of the right side of (6.87) by the initial density r_A of A particles gives an upper bound for $\rho_A(t)$. Substituting in $t = 2t_1$ and $\lambda < C_{19}/2$, this implies (6.79) for $d > 2$. ∎

In the case $d = 2$, the bound (6.86) is not good enough to give (6.79). Before proceeding to improve (6.86), we pause briefly to point out some of the complications one encounters in $d = 2$. The choice of $\delta > 1/2$ (and hence $\delta' > 0$) was made in Lemma 6.6. It can be checked that if the right side of (6.68) were replaced by the term $\exp\{C_{13}(r_A, r_B)\, g_d(t_1)\}$, $C_{13}(r_A, r_B) > 0$, then $\delta > 1$ could be chosen for $d = 2$ as well. The analog of (6.87) would then hold in $d = 2$ as well but with some function $C_{19}(r_A, r_B)$ substituted for the product $C_{19}(r_B - r_A)$. The main complication in $d = 2$ arises from the need to consider cubes D_{R_1} with R_1 large, if (6.69) and (6.71) in Lemma 6.6 (with $l = C_{12}(r_B - r_A)/2$) are to hold. For R_1 large, L in (6.26) will be large; this restricts the number of iterations of Lemma 6.5 available for (6.73) in Lemma 6.6, which in turn may force us to choose $\delta < 1$ if the bound on $\mathfrak{N}^A_{2t_1}(t_1)$ on the left side of (6.68) is to hold. For $d > 2$ on the other hand, (6.73) easily holds no matter what the choice of L. Note that this whole procedure is meaningless in $d = 1$, since for the bound on the right side of (6.68) (even in weakened form) to hold, one would need to plug $s_1 = R_1 \sim \sqrt{t_1}$ into (6.60), which would leave no time for iteration.

Part II—Conclusion for $d = 2$. The bounds given in (6.85) over $E^{\delta}(t_1)$ suffice for $d = 2$ as well; it is the set $(E^{\delta}(t_1))^c$ in (6.86) which presents difficulties. We take advantage of (6.85) by decomposing $\rho_A(s)$ into $\rho_{A_1}(s)$, $\rho_{A_2}(s)$, with $\rho_A(s) = \rho_{A_1}(s) + \rho_{A_2}(s)$, so that

$$\begin{aligned} \rho_{A_1}(s) &= \text{density of } A \text{ particles at time } s \text{ which have remained in} \\ &\qquad y + D_{t_1^{\delta}} \text{ over } [t_1, (2t_1) \wedge s], \\ \rho_{A_2}(s) &= \text{density of } A \text{ particles at time } s \text{ which have not remained in} \\ &\qquad y + D_{t_1^{\delta}} \text{ over } [t_1, (2t_1) \wedge s], \end{aligned} \tag{6.88}$$

where in each case, y is the position of the particle at time t_1. Call the two classes of A particles referred to in (6.88), A_1 particles and A_2 particles, respectively. On account of (6.85),

$$\rho_{A_1}(2t_1) \leqslant r_A \exp\{-C_{18}(r_B - r_A)\, g_d(t_1)\}. \tag{6.89}$$

On the other hand, $\xi_s^{A_2}$, $s \geqslant 2t_1$, is dominated by $\eta_s^{A_2}$, where as always, η is the process whose random walks never interact and which is coupled to ξ. Since the A_2 particles in η move independently, the probability given in (6.86) is small enough to give us the analog for A_2 particles of (6.68) in Lemma 6.6 for any $\delta_1 > 1$, if we delay our procedure by t_1. We will be able to use this and (6.89) to obtain the analog of (6.83), but for δ_1 instead of only for some $\delta > 1/2$. The remainder of the reasoning through (6.87) follows exactly as before, but where we can now employ the improved bound given in (6.86) to obtain (6.79). Before proceeding with the argument, we note as in the beginning of the proof that it will be convenient to translate $\mathbb{Z}^d$, this time by $Y^0_{2t_1}$, when considering the interaction of Y^0 and ξ on $s \in [2t_1, 3t_1]$. For $\hat{Y}^0_s = Y^0_s - Y^0_{2t_1}$, $\hat{Y}^0_{2t_1} = 0$. As before, $\hat{\xi} = \xi - Y^0_{2t_1}$ has the same distribution as ξ, and $\hat{\eta}^{A_2} = \eta^{A_2} - Y^0_{2t_1}$ the same distribution as η^{A_2}. We drop the superscript " ^ " from $\hat{Y}$, $\hat{\xi}$, and $\hat{\eta}^{A_2}$.

We first note that since $\rho_{A_1}(s)$ is decreasing in s, the bound in (6.89) holds at $s = 3t_1$ as well. So for large t and fixed δ_1,

$$P[\mathfrak{D}^{A_1}_{3t_1^{\delta_1}}(3t_1; \xi) \neq 0] \leqslant 3^d t_1^{\delta_1 d} r_A \exp\{-C_{18}(r_B - r_A)\, g_d(t_1)\}. \tag{6.90}$$

A random walk which is in $D_{2t_1^{\delta_1}}$ at some time in $[2t_1, 3t_1]$ will typically (with probability close to 1) still be in $D_{3t_1^{\delta_1}}$ at time $3t_1$. The reasoning is standard and was given after (6.24). So (6.90) implies that for large t,

$$\begin{aligned} &P[\mathfrak{D}^{A_1}_{2t_1^{\delta_1}}(s; \xi) \neq 0 \text{ for some } s \in [2t_1, 3t_1]] \\ &\qquad \leqslant \exp\{-C_{20}(r_B - r_A)\, g_d(t_1)\}, \end{aligned} \tag{6.91}$$

where $C_{20} > 0$.

On account of (6.86) and the independence of the motion of particles in η, η^{A_2}, $2t_1 \leqslant s \leqslant 3t_1$, has a Poisson measure with density at most $4dr_A \exp\{-c_6' t^{\delta'}\}$, $\delta' > 0$. It is therefore not difficult to show that

$$\begin{aligned} &P[\#(A_2 \text{ particles in } \xi \text{ intersecting } D_{2t_1^{\delta_1}} \text{ over } [2t_1, 3t_1]) \geqslant C_{14}(r_B - r_A)\, g_d(t_1)/2] \\ &\quad \leqslant P[\#(A_2 \text{ particles in } \eta \text{ intersecting } D_{2t_1^{\delta_1}} \text{ over } [2t_1, 3t_1]) \geqslant C_{14}(r_B - r_A)\, g_d(t_1)/2] \\ &\quad \leqslant \exp\{-C_{21}(r_B - r_A)\, g_d(t_1)\} \end{aligned} \tag{6.92}$$

for C_{14} as chosen earlier and $C_{21} > 0$. One can for instance make the same substitution for the density as used for (6.24) in the proof of Lemma 6.3 to obtain a comparable upper bound on the number of A_2 particles in $D_{2t_1^{\delta_1}}$ at time $3t_1$. One can then once again reason as after (6.24) to extend this bound to (6.92).

Together, (6.91) and (6.92) show that

$$\begin{aligned} &P[\mathfrak{N}^{A}_{2t_1^{\delta_1}}(t_1) \geqslant C_{14}(r_B - r_A)\, g_d(t_1)/2] \\ &\quad = P[\#(A \text{ particles in } \xi \text{ intersecting } D_{2t_1^{\delta_1}} \text{ over } [2t_1, 3t_1]) \geqslant C_{14}(r_B - r_A)\, g_d(t_1)/2] \\ &\quad \leqslant \exp\{-C_{22}(r_B - r_A)\, g_d(t_1)\}, \end{aligned} \tag{6.93}$$

$C_{22} > 0$. (6.93) corresponds to (6.83), the only difference being the time interval $[2t_1, 3t_1]$ here instead of $[t_1, 2t_1]$. As in (6.84), this gives an upper bound on the number of B particles $\mathfrak{N}^{B}_{2t_1^{\delta_1}}(t_1)$ remaining in $D_{2t_1^{\delta_1}}$ over $[2t_1, 3t_1]$ which can be annihilated. As mentioned earlier, the analog of (6.82) which gives a lower bound on $\hat{\mathfrak{L}}^{\delta_1}_1(t_1)$ holds. Comparing $\hat{\mathfrak{L}}^{\delta_1}_1(t_1)$ and $\mathfrak{N}^{B}_{2t_1^{\delta_1}}(t_1)$, we see that the analog of (6.85) for $\hat{\mathscr{K}}^{\delta_1}(t_1)$ and $\hat{E}^{\delta_1}(t_1)$ therefore holds. That is, on $\hat{E}^{\delta_1}(t_1)$, Y^0 typically meets at least one B particle in ξ during $[2t_1, 3t_1]$. On the other hand, (6.86) clearly holds if $\hat{E}^{\delta_1}(t_1)$ is substituted for $E^{\delta}(t_1)$. Here, however, $\delta_1 > 1$, and so $\delta_1' > 1$. Applying these versions of (6.85) and (6.86), we see that the analog of (6.87) holds in $d = 2$ as well. Substituting $t = 3t_1$, this implies (6.79) for $d = 2$ with $\lambda < (C_{17} \wedge C_{22})/3$. ∎

7. Upper Bounds for Unequal Densities, $d = 1$

In this section we give upper bounds on $\rho_A(t)$ for $d = 1$. Here, the argument is different than that for $d \geqslant 2$. For $d = 1$, there are not sufficiently many B particles close enough to a given A particle to apply the iteration scheme used in Lemmas 6.4–6.5. On the other hand, one can now use the linearity of $\mathbb{Z}$. In particular, particles cannot pass by one another without meeting. The different nature of the arguments for $d = 1$ and $d > 1$ is not too surprising given the different dependence on the initial densities r_A and r_B.

Here, we modify notation somewhat and let

$$\mathfrak{D}_x = (\#B \text{ particles}) - (\#A \text{ particles}) \text{ at time } 0- \text{ in } [1, x] \tag{7.1a}$$

if $x > 0$; for $x \leqslant 0$, we let

$$\mathfrak{D}_x = (\#B \text{ particles}) - (\#A \text{ particles}) \text{ at time } 0- \text{ in } [x, 0]. \tag{7.1b}$$

Our first lemma gives a simple bound on how small $\mathfrak{D}_x$ can get.

Lemma 7.1. Assume that A and B particles are initially distributed over $\mathbb{Z}$ according to (1.1) with $r_A < r_B$. Then for $x \geqslant 0$,

$$P[\mathfrak{D}_y \leqslant \tfrac{1}{2}(r_B - r_A)x \text{ for some } y,\ |y| \geqslant x] \\ \leqslant C_1 \exp\{-(r_B - r_A)^2 x/24 r_B\}, \tag{7.2}$$

where C_1 depends on r_A, r_B, and

$$P[\mathfrak{D}_y \leqslant -N \text{ for some } y] \leqslant 2\left(\frac{r_A}{r_B}\right)^N \leqslant 2e^{-N(r_B - r_A)/r_B}. \tag{7.3}$$

Proof. By Lemma 5.1,

$$P[\mathfrak{D}_y \leqslant \tfrac{1}{2}(r_B - r_A)|y|] \leqslant \exp\{-(r_B - r_A)^2 |y|/24 r_B\}. \tag{7.4}$$

Summing up these probabilities over $|y| \geqslant x$ gives

$$\begin{aligned} &P[\mathfrak{D}_y \leqslant \tfrac{1}{2}(r_B - r_A)x \text{ for some } y,\ |y| \geqslant x] \\ &\quad \leqslant P[\mathfrak{D}_y \leqslant \tfrac{1}{2}(r_B - r_A)|y| \text{ for some } y,\ |y| \geqslant x] \\ &\quad \leqslant 2 \sum_{k=x}^{\infty} \exp\{-(r_B - r_A)^2 k/24 r_B\} \\ &\quad = 2\exp\{-(r_B - r_A)^2 x/24 r_B\}/(1 - \exp\{-(r_B - r_A)^2/24 r_B\}); \end{aligned}$$

this implies (7.2). One can use moment generating functions as in Lemma 5.1 to obtain estimates like (7.3). Instead, we proceed as follows. It is easy to check that the random variable

$$M_y = (r_A/r_B)^{\mathfrak{D}_y}$$

is a positive martingale in the parameter $y \geqslant 0$, since each particle present at time $0-$ has probability $r_A/(r_A + r_B)$ of being an A particle and $r_B/(r_A + r_B)$ of being a B particle; $M_0 = 1$. Stopping M_y at

$$T_n = \min\{y > 0 \colon \mathfrak{D}_y \leqslant -N\} \wedge n$$

and applying Chebyshev's inequality gives

$$P[M_{T_n} \leqslant -N] \leqslant \left(\frac{r_A}{r_B}\right)^N E[M_{T_n}] = \left(\frac{r_A}{r_B}\right)^N.$$

Letting $n \to \infty$, we obtain

$$P[\mathfrak{D}_y \leqslant -N \text{ for some } y > 0] \leqslant \lim_{n \to \infty} P[M_{T_n} \leqslant -N] \leqslant \left(\frac{r_A}{r_B}\right)^N.$$

Including negative values of y gives the factor of 2. ∎

We now sketch the main idea behind Proposition 6. We will show that except on a set of exponentially small probability, there will always be a pair of B particles, one starting from $[1, \delta\sqrt{t}]$ and the other from $[-\delta\sqrt{t}, 0]$ ($\delta > 0$ to be chosen later), which meet by time t. An A-particle starting at 0 must have already, in this case, met some (other) B-particle by time t.

To be more specific, we introduce the following notation. Let

$$F_1 = \{\mathfrak{D}_y > \tfrac{1}{2}(r_B - r_A)\,\delta\sqrt{t} \text{ for all } y, |y| \geqslant \delta\sqrt{t}\}$$

and

$$F_2 = \{\mathfrak{D}_y > -\tfrac{1}{8}(r_B - r_A)\,\delta\sqrt{t} \text{ for all } y\}.$$

Set $F = F_1 \cap F_2$. By Lemma 7.1,

$$\begin{aligned} P[F^c] &\leqslant P[F_1^c] + P[F_2^c] \\ &\leqslant C_2 \exp\{-(r_B - r_A)^2\,\delta\sqrt{t}/24r_B\}, \end{aligned} \tag{7.5}$$

where C_2 depends on r_A, r_B. On account of (7.5), we will be able to restrict our attention to the set F.

We begin by constructing processes X_s^k, $k = -K^-, \dots, -1, 1, \dots, K^+$, corresponding to the motion of B particles with

$$K^{\pm} \geqslant [\tfrac{1}{2}(r_B - r_A)\,\delta\sqrt{t}] \quad \text{on } F. \tag{7.6}$$

Here and later on, we will only specify the behavior for $k > 0$, it being understood that $k < 0$ is defined correspondingly after reflecting about 0. The processes X^k will be random walks with drift to the left on an appropriate set $G_1 \subset \Omega$. We begin by setting

$$X_0^k = x_k = \min\{x\colon 1 \leqslant x \leqslant \delta\sqrt{t}, \mathfrak{D}_y \geqslant k, \forall y \geqslant x\}. \tag{7.7}$$

K^+ is to denote the largest such index where the inequality is attained. Clearly, there are at least as many B particles at x as indices k with $x_k = x$. We initially tie the motion of X^k to that of the corresponding B particle. We call such B particles "marked." Note that on F_1, (7.6) holds. We will shortly continue X^k so that

$$X_s^k = Y_s^k - Z_s^k \qquad \text{for} \quad s \leqslant \tau^k, \tag{7.8}$$

where Y^k is a random walk with $Y_{0-}^k = X_{0-}^k = x_k$, and Z^k is an increasing process on G_1 with $Z_{0-}^k = 0$. Y^k for different k will be independent of one another. τ^k will be the time at which X^k disappears.

To do so, we first consider the remaining (= unmarked) B particles, and all the A particles. Order the initial positions ${}^{A}q_i$ and ${}^{B}q_j$ of these A and B particles so that for $i<i'$, $j<j'$,

$$ {}^{A}q_i \leqslant {}^{A}q_{i'}, \qquad {}^{B}q_j \leqslant {}^{B}q_{j'}. \tag{7.9} $$

Introduce processes ${}^{A}Q_s^i$, ${}^{B}Q_s^j$, with ${}^{A}Q_0^i = {}^{A}q_i$, ${}^{B}Q_0^j = {}^{B}q_j$, which evolve according to the corresponding A and B particles. We will find it convenient to use the convention that as long as the particles involved survive, the analog of (7.9) holds for A particles, that is, for $i<i'$,

$$ {}^{A}Q_s^i \leqslant {}^{A}Q_s^{i'}. \tag{7.10} $$

Of course, this ordering just involves bookkeeping since A particles are indistinguishable, and so does not affect the behavior of our underlying process ξ. (With a little more work, one could avoid this ordering.) Denote by ${}^{A}\sigma_i$, ${}^{B}\sigma_j$ the times at which ${}^{A}Q^i$ and ${}^{B}Q^j$ are annihilated.

In Proposition 6, we will show that with high probability, B particles starting from $[1, \infty)$ and from $(-\infty, 0]$ meet by time t. Any A particle starting at 0 must therefore be annihilated by then. We find it convenient to let Z_s^0 be a designated A particle in ξ with $Z_0^0 = 0$, and G_1 the event that it survives up until time t. To demonstrate the proposition, it will be enough to obtain good upper bounds on $P[G_1]$. Since the evolution of A particles is assumed to be order preserving, no B particle from either side of Z^0 can hit an A particle from the other side without first hitting Z^0. So under G_1, the systems of particles starting in $[1, \infty)$ and in $(-\infty, 0]$ can be treated as separate noninteracting systems. We use this for the purpose of constructing our processes X^k below and for Lemma 7.3. The processes X^k and X^{-k} will correspond to the motion of B particles. On account of (7.8) and its analog for negative indices, X^k and X^{-k} will behave on G_1 like independent random walks with drift toward one another. ($Z^{\pm k}$ will arise from the substitution of another B particle after each annihilation.) On F the number of such pairs is large; the probability that no such pair meets by time t should therefore be small. This reasoning, which resembles a proof by contradiction, will enable us to show that $P[G_1]$ is in fact exponentially small.

To define X^k, we first establish a relationship between $\{{}^{A}Q^i\}$ and $\{{}^{B}Q^j\}$ on G_1. We let $\varphi_{0-}(j) = i$ denote the index of the first A particle which at time $0-$ lies to the right of (maybe at the same spot as) the jth unmarked B-particle and $(i-1)$st A-particle. So all unmarked B-particles are initially "associated" with A-particles which are to their right. Let $\psi_{0-}(i) = j$ denote the inverse of φ_{0-}, where we set $\psi_{0-}(i) = -\infty$ if there is no B particle associated with the ith A-particle. We continue $\psi_s(i) = j$,

$s > 0$, on G_1 so that $\psi_s(i)$ remains constant until either the A or the associated B particle is annihilated. For $s \geqslant {}^A\sigma_i$, set $\psi_s(i) = \infty$. More importantly, at $s = {}^B\sigma_j$, do the following under ${}^B\sigma_j < {}^A\sigma_i$, in which case ${}^BQ^j$ annihilates some ${}^AQ^{i'}$ with $i' \neq i$: set $\psi_s(i) = \psi_{s-}(i')$. That is, ${}^AQ^i$ adopts as its associate the former associate of the A particle just annihilated by ${}^AQ^i$'s own former associate ${}^BQ^j$. Since ${}^BQ^j_{s-} < {}^AQ^i_{s-}$, it follows that $i' < i$, and so

$$ {}^BQ^{j'}_s < {}^AQ^{i'}_s = {}^BQ^j_s < {}^AQ^i_s \tag{7.11} $$

where $j' = \psi_{s-}(i')$. The new associated B-particle (for $\psi_s(i) \neq \infty$) therefore also lies to the right of ${}^AQ^i_s$. Continuing in this manner, one defines ψ_s for all s.

We define X^k_s on G_1 to be the position of the corresponding marked B particle (from time $0-$) until the time $s = T^k_1 = {}^A\sigma_i$ at which the particle is annihilated by the ith A-particle. If $j = \psi_{s-}(i) \neq -\infty$, then we identify X^k with ${}^BQ^j$ starting at time s, until that particle is in turn annihilated. Label ${}^BQ^j$ starting at time s as marked. If $\psi_{s-}(i) = -\infty$, then we set $\tau^k = s$, at which time X^k disappears. One can construct X^k on G_1 for all s by using this procedure at the times $T^k_1, T^k_2,\dots$ at which the corresponding B-particles are annihilated. On account of (7.11), it is easy to see that

$$ X^k_s < X^k_{s-} \quad \text{at} \quad s = T^k_l, \qquad l = 1, 2,\dots. \tag{7.12} $$

One can therefore represent X^k as in (7.8), where

$$ Z^k_s - Z^k_{s-} = -(X^k_s - X^k_{s-}). \tag{7.13} $$

Clearly, Z^k_s is increasing in s. Y^k_s is the random walk motion of X^k between these times and is independent of the other random walks corresponding to different values of k. We can also represent X^k as in (7.8) on G^c_1, except that we now have no control over Z^k since particles from $(-\infty, 0]$ may become involved. The nature of X^k on G^c_1 will not be important. We will find it convenient to always extend Y^k (still as a random walk) for all s, including $s > \tau^k$.

To demonstrate Proposition 6, we need two observations, which we state as lemmas. We let $\mathfrak{N}^+_s$ be the number of unassociated A particles on G_1 at time s (particles still surviving with $\psi_s = -\infty$) with initial positions to the right of 0; $\mathfrak{N}^-_s$ is defined analogously, but for the system of particles starting to the left of 0. Also, let $\mathscr{A}^+_s$ ($\mathscr{A}^-_s$) be the number of processes X^k with $k > 0$ ($k < 0$) which have disappeared by time s. By $\mathfrak{D}^A_x$ (resp., ${}^M\mathfrak{D}^B_x$, ${}^U\mathfrak{D}^B_x$, ${}^T\mathfrak{D}^B_x = {}^M\mathfrak{D}^B_x + {}^UD^B_x$) we will mean the number of A particles (resp., the numbers of marked and unmarked, and total number of B particles) initially in $[1, x]$, $x > 0$.

Lemma 7.2. On $F \cap G_1$,

$$\mathfrak{N}_{0-}^{\pm} \leqslant \tfrac{1}{8}(r_B - r_A)\,\delta\sqrt{t}.$$

Proof. We give the argument for $\mathfrak{N}_{0-}^{+}$, since the case $\mathfrak{N}_{0-}^{-}$ is analogous. Set $N = [\frac{1}{8}(r_B - r_A)\,\delta\sqrt{t}]$. We assume that there are unassociated A particles at $y_1 \leqslant \cdots \leqslant y_N$. Let $y_{N+1} \geqslant y_N$ be the site of an A particle; we wish to show this A particle is associated with an unmarked B particle to its left. Note that

$$\mathfrak{D}_{y_{N+1}} \geqslant -N \quad \text{on } F. \tag{7.14}$$

We assume that in fact the A particle at y_{N+1} is unassociated, and show this gets us into trouble. Consider the two cases where (1) ${}^{M}\mathfrak{D}^{B}_{y_{N+1}} = 0$ and (2) ${}^{M}\mathfrak{D}^{B}_{y_{N+1}} > 0$. Under (1), (7.14) implies that

$${}^{U}\mathfrak{D}^{B}_{y_{N+1}} = {}^{T}\mathfrak{D}^{B}_{y_{N+1}} = \mathfrak{D}^{A}_{y_{N+1}} + \mathfrak{D}_{y_{N+1}} \geqslant \mathfrak{D}^{A}_{y_{N+1}} - N \tag{7.15}$$

on F. On the other hand, at most one B particle is associated with each of the other $(\mathfrak{D}^{A}_{y_{N+1}} - (N+1))$ A-particles to the left of y_{N+1}, and these $N+1$ unattached A-particles contribute no B particles. Moreover, each unmarked B particle to the left of y_{N+1} must be associated with an A particle to the left of y_{N+1}. One therefore also has

$${}^{U}\mathfrak{D}^{B}_{y_{N+1}} \leqslant \mathfrak{D}^{A}_{y_{N+1}} - (N+1). \tag{7.16}$$

(7.16) contradicts (7.15); the particle at y_{N+1} must therefore be associated with a B particle to its left in case (1).

Under (2),

$$\mathfrak{D}_{y_{N+1}} \geqslant {}^{M}\mathfrak{D}^{B}_{y_{N+1}} \tag{7.17}$$

because of our definition of marked particles in (7.7). (This is the reason for the proviso "$\forall y \geqslant x$" there.) Consequently,

$${}^{U}\mathfrak{D}^{B}_{y_{N+1}} - \mathfrak{D}^{A}_{y_{N+1}} = \mathfrak{D}_{y_{N+1}} - {}^{M}\mathfrak{D}^{B}_{y_{N+1}} \geqslant 0. \tag{7.18}$$

So there are at least as many unmarked B particles as A particles to the left of y_{N+1}; the A particle at y_{N+1} must therefore be associated to some B particle to its left in case (2) as well. ▌

Lemma 7.3. On $F \cap G_1$,

$$\mathfrak{N}_{0-}^{\pm} - \mathfrak{N}_{s}^{\pm} \geqslant \mathscr{A}_{s}^{\pm} \qquad \text{for all } s. \tag{7.19}$$

Proof. Based on our construction, on G_1, X^k can only disappear when it hits an unassociated A particle starting from the same side of 0. So by time

s, $\mathscr{A}_s^\pm$ unassociated A particles have been annihilated due to interaction with marked B particles. On F, $\mathfrak{N}_s^\pm$ is finite, and can never increase. For when ${}^AQ^i$ is stripped of its associated B particle at time s, ${}^BQ^j$ with $j=\psi_{s-}(i)$, then either (1) it receives another B particle ${}^BQ^{j'}$ in return or (2) the A particle ${}^AQ^{i'}$ annihilated by ${}^BQ^j$ was unassociated. In the first case there is no change in unassociated A particles, whereas in the second case one is created and one is destroyed. Consequently,

$$\mathfrak{N}_{0-}^\pm - \mathfrak{N}_s^\pm \geqslant \mathscr{A}_s^\pm \qquad \text{for all } s. \quad \blacksquare$$

One can in fact replace the inequality in (7.19) with an equality. If $\mathfrak{D}_x^A \to \infty$ as $x \to \infty$, then all B particles are either marked or attached to some A particle, and so $\mathfrak{N}_s^\pm$ cannot decrease without an unattached A particle hitting some X^k. In the exceptional case where $\mathfrak{D}_x^A$ stays bounded, some further thought shows this is still true. We will not use this direction, however.

The following is a direct consequence of Lemmas 7.2 and 7.3.

Corollary 1. On $F \cap G_1$,

$$\mathscr{A}_t^\pm \leqslant \tfrac{1}{8}(r_B - r_A)\,\delta\sqrt{t}.$$

Let G_2 be the event that for at least $\frac{1}{4}(r_B - r_A)\,\delta\sqrt{t}+1$ values of $1 \leqslant k \leqslant K \equiv (K^+ \wedge K^-)$, the random walks Y^k defined in (7.8) and (7.13) satisfy

$$Y_s^k = Y_s^{-k} \qquad \text{for some } s \in [0, t]. \tag{7.20}$$

We will show in Proposition 6 that conditioned on F, G_2 occurs with high probability. Corollary 1 above puts $\frac{1}{4}(r_B - r_A)\,\delta\sqrt{t}$ as the upper bound on the number of pairs of particles X^k and X^{-k} at least one of which is annihilated. Since X^k, X^{-k} are coupled to Y^k, Y^{-k} but with an extra component of drift toward one another, under G_2 some pair X^k, X^{-k} must also meet by time t. Any A particles starting in between this pair must be annihilated by time t. On account of the high probability of F given in Lemma 7.1, this implies (7.21).

Proposition 6. Assume that A and B particles are initially distributed over $\mathbb{Z}$ according to (1.1) with $r_A < r_B$. Then

$$\rho_A(t) \leqslant \exp\{-\lambda((r_B - r_A)^2/r_B)\sqrt{t}\} \tag{7.21}$$

for appropriate $\lambda > 0$ and large enough t.

Proof. For $1 \leqslant k \leqslant K$, $Y_s^k - Y_s^{-k}$ is a rate-2 simple random walk with

$$Y_0^k - Y_0^{-k} \leqslant 2\delta\sqrt{t}. \tag{7.22}$$

It is a simple consequence of the central limit theorem and the reflection principle that for δ small enough and t not too small,

$$P[Y_s^k = Y_s^{-k} \text{ for some } s \in [0, t]] \geqslant 3/4$$

for each k. Also, by (7.6),

$$K \geqslant [\tfrac{1}{2}(r_B - r_A)\,\delta\sqrt{t}] \quad \text{on } F.$$

It therefore follows from Corollary 1 of Lemma 4.3, that

$$P[G_2^c \mid F] \leqslant \exp\{-C_3(r_B - r_A)\,\delta\sqrt{t}\} \tag{7.23}$$

for appropriate $C_3 > 0$. Together with (7.5), this shows that

$$P[F^c \cup G_2^c] \leqslant \exp\{-C_4\delta((r_B - r_A)^2/r_B)\sqrt{t}\} \tag{7.24}$$

for appropriate $C_4 > 0$.

As discussed after (7.10), to demonstrate (7.21) it suffices to obtain upper bounds on $P[G_1]$, where G_1 is the event that a designated A particle Z^0 with $Z_0^0 = 0$ survives until time t. We may write

$$G_1 = \{X_s^k \neq X_s^{-k}\ \forall s, \forall k\} \cap G_1, \tag{7.25}$$

where $s \in [0, t]$ and $1 \leqslant k \leqslant K$ are understood $(G_1 \subset \{\cdot\})$. This is

$$\subset (\{X_s^k \neq X_s^{-k}\ \forall s \text{ for } \tau^k > t, \tau^{-k} > t\} \cap F \cap G_1 \cap G_2) \cup F^c \cup G_2^c. \tag{7.26}$$

We will show that the quantity inside the parentheses is void by following the outline sketched after (7.20).

Recall from (7.8) that on G_1,

$$X_s^k - X_s^{-k} \leqslant Y_s^k - Y_s^{-k}, \qquad \text{for all } s.$$

The quantity inside the parentheses in (7.26) is therefore contained in

$$(\{Y_s^k \neq Y_s^{-k}\ \forall s, \tau^k > t, \tau^{-k} > t\} \cap F \cap G_1) \cap G_2. \tag{7.27}$$

Also, by Corollary 1 to Lemma 7.3, on $F \cap G_1$,

$$|\{k: \tau^k > t, \tau^{-k} > t\}^c| \leqslant \tfrac{1}{4}(r_B - r_A)\delta\sqrt{t}.$$

So Y^k and Y^{-k} can meet for only $\frac{1}{4}(r_B - r_A)\,\delta\sqrt{t}$ many pairs. By the definition of G_2, this implies that the intersection of $(\cdot)$ with G_2 in (7.27) is void. So the quantity in the parentheses in (7.26) is also void.

It therefore follows from (7.25)–(7.26) that

$$P[G_1] \leqslant P[F^c \cup G_2^c]. \tag{7.28}$$

By (7.24), (7.28) is

$$\leqslant \exp\{-C_4 \delta((r_B - r_A)^2/r_B)\sqrt{t}\}.$$

This last term can be rewritten as in (7.21). ∎

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